GROUP - VELOCITY
OF ATMOSPHERIC WAVES

by

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B.A., THE AMERICAN UNIVERSITY OF BEIRUT (1934)

SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF SCIENCE at the MASSACHUSETTS INSTITUTE OF TECHNOLOGY (1946)

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ACKNOWLEDGMENT

The author wishes to express his deep respect and gratitude to his advisor Professor B.Haasmdit, for his guidance in the development of this thesis, and for the corrections which he made.

He is much indebted to Professor H.G.Houghton, who, by his scientific far sightedness, was the first to suggest the subject of this thesis to the author. The author was led to his ideas about the lack of peridicity in cyclones mainly through the discussion of this subject with Professor Houghton.

Finally, the author wishes to thank Mr. A.N. Dingle of M.I.T. for his help in correcting the language.
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CHAPTER I

INTRODUCTION

1- The Concept of Group-Velocity

It has often been remarked that, when a group of waves advances into still water, the velocity of the group as a whole is less than that of the individual waves of which it is composed. If attention be fixed on a particular wave, it is seen to advance through the group, gradually dying away as it approaches its anterior limit, whilst its former place in the group is occupied in succession by other waves which appear to have come forward from outside the group.

We are led, thus, to the concept of the group-velocity as distinct from the wave velocity. The group-velocity being that of the group as a whole, and the wave velocity that of the individual waves which constitute it. In modern physics the wave velocity is usually called the "phase-velocity", and we shall use this nomenclature throughout in this thesis in order to avoid any misunderstanding that might arise from mixing the two terms.

The concept of group-velocity has found many important applications in different branches of Physics, such as the propagation of a wireless impulse, the electromagnetic theory of light, wave-mechanics, the new quantum theory...etc. It has also been given a good deal of attention in hydrodynamics and as a matter of fact, the theory was first initiated to
to explain phenomena relating to the propagation of waves on the surface of deep water.

From a physical point of view the group-velocity is perhaps even more important and significant than the phase-velocity. The latter may be greater or less than the former, and it is even possible to have media in which it would have the opposite direction, i.e. a disturbance might be propagated outwards from a center in the form of a group, whilst the individual waves composing the group were themselves travelling backwards coming into existence at the front, and dying out as they approach the rear. Moreover even in the most familiar phenomena of Acoustics and Optics the phase-velocity is of importance chiefly so far as if coincides with the group-velocity.

2- Historical

The first scientific paper written on group-velocity was that of Scott Russell, "Report on Waves", (Reference 19), in which he described the phenomenon as observed in water. However, it seems that Stokes (1875) was first to show that the phenomenon was capable of being treated analytically, by showing that when two infinite trains of waves, of equal amplitude and nearly equal wave-lengths are superposed, we obtain an infinite succession of wave-groups, each of which advances with the half-phase-velocity. None of the groups maintains its outline constant for any interval of time, however short; but the whole disturbance periodically returns to the
configuration it had at any particular instant if the periods of the superposed trains be commensurable, and the effect is the same as if each group had moved forward in the interval, without change of shape, at the half-phase-velocity.

A most important contribution to the theory of group-velocity was then made by Osborne Reynolds in Nature, Aug. 25, 1877, (Reference17), where he gave a dynamical explanation of the fact that the regular part of a large group of waves of equal wave-lengths advances with only half the velocity of the separate waves; He proved that the energy propagated across a plane, when a regular train of waves is passing, is just sufficient to feed a regular procession of waves, travelling with half the corresponding phase-velocity.

Lord Rayleigh (1881) pointed out in his article "On Progressive Waves" (Ref.16) that the theorem given by Osborne Reynolds for the case of a regular procession of water-waves could be extended to apply to all kinds of waves. The same thing was also shown by Gouy (1889), (Ref.5), who applied the theory to the propagation of light. The theory was later developed by H. Lamb and George Green (1909), (Refs.6 and 25)

**Analytical Considerations**

3-Case of Two Simple Harmonic Trains

In general when waves are started by a local disturbance, such as for example, the dropping of a stone into a pond, the successive waves have different lengths and are propagated with different velocities. Let us examine the
the phenomena that arise from the simultaneous motion in the
same medium and in the same direction of two simple harmonic
trains of waves of the same amplitude and slightly different
wave-lengths.

We may write for the elevation at any point:
\[ \eta = a \sin (m \chi - m \ell) + a \sin (m' \chi - m' \ell) \]
\[ = 2a \cos \frac{1}{2} [(m+m') \chi - (m+m') \ell] \cos \frac{1}{2} [(m+m') \chi - (m+m') \ell] \]
\[ = 2a \cos \frac{1}{2} [(m+m') \chi - (m+m') \ell] \]
\[ \ldots (v) \]

If \( m = m' \) nearly, \( (m+m') \chi \) varies with \( \chi \) much
more slowly than does \( (m+m') \ell \), so that it is convenient at any
instant to regard the equation as representing a sineous curve
\[ \eta = 2a \cos \frac{1}{2} [(m+m') \chi - (m+m') \ell] \]
and multiplying the ordinate by:
\[ 2a \cos \frac{1}{2} [(m+m') \chi - (m+m') \ell] \]

Hence the resulting curve represents a train of waves
whose amplitude
\[ 2a \cos \frac{1}{2} [(m+m') \chi - (m+m') \ell] \]
is periodic, varying very slowly with \( \chi \) between 0 and 2a.
The profile of this train will be a group of sinuosities of
amplitude gradually increasing from zero to 2a and then
decreasing to zero followed by a succession of equal groups.
The appearance on water will be that of alternate groups of
waves separated by intervals of nearly still water. The motion
of each group is then sensibly independent of the others.

Since the distance between the center of two
successive groups is \( 2\pi/(m-m') \), and the time occupied by
the system in shifting through this space is \( 2\pi/(n-n') \),
the group-velocity (W, say) is:
\[ W = \frac{m - m'}{m - m'}, \quad \text{or} \]
\[ W = \frac{dv}{dm} \quad \ldots \ldots \quad (2.1) \]

when the difference of the wave-lengths of the original trains is small.

Since the velocity of a single wave (the phase-velocity) is
\[ V = \frac{m}{m'} \quad \ldots \ldots \ldots \ldots \ldots \quad (3.1) \]
therefore:
\[ W = \frac{d(-mV)}{dm} \]
\[ \text{or:} \quad W = V + m \frac{dV}{dm} \quad \ldots \ldots \ldots \ldots \quad (4.1) \]
or if \( \lambda \) be the wave-length, \( \frac{2\pi}{m} \),
\[ W = V - \lambda \frac{dV}{d\lambda} \quad \ldots \ldots \ldots \ldots \quad (5.1) \]

This result holds for any case of waves travelling through a uniform medium.

4- Case of an Infinite Number of Waves

The theory of group-velocity has been treated in a more general manner by Lord Rayleigh (Ref.13).

A disturbance travelling in one direction can be resolved by Fourier's theorem into infinite trains of waves of harmonic type and of various amplitudes and wave-lengths. And the only case in which one can expect a simple result is that in which a considerable number of consecutive waves are sensibly of a given harmonic type, though the wave-length and amplitude may vary within moderate limits at points whose distances amount to a large multiple of \( \lambda \).
Assuming that the complete expression by Fourier's series involves only wave-lengths which differ but little from one another, we may write:

\[ \eta = a_1 \sin [(m + \delta m) x - (n + \delta n) t + \xi] + a_2 \sin [(m + 2\delta m) x - (n + 2\delta n) t + \xi] + \ldots \]

\[ = \sin (m x - nt) \sum a_r \cos (r \delta m - t \delta n + \xi) + \cos (m x - nt) \sum a_r \sin (r \delta m - t \delta n + \xi) + \ldots \]  

(6)

Also, by hypothesis: \( \frac{\delta n}{\delta m} = \frac{\delta m}{\delta m} = \ldots = \frac{d\eta}{dm} \)

and the first term in the expression for \( \eta \) represents a simple train of type \( \sin (mx - nt) \) with varying amplitude

\[ \sum a_r \cos (r \delta m - t \delta n + \xi) \]

and the amplitude itself is propagated as a wave with the velocity

\[ w = \frac{d\eta}{dm} \]

and similarly the second term. Hence we arrive at the idea of groups of waves of a more general kind, but the velocity of propagation is given by the same formula as in the special case of two waves discussed above.

Another derivation of equation 5 was given by Lamb (Ref. 25), which is as follows:

In a medium such as we are considering, where the wave-velocity varies with the frequency, a limited initial disturbance gives rise in general to a wave-system in which the different wave-lengths, travelling with different velocities, are gradually sorted out. If we regard the wave-
length \( \lambda \) as a function of \( x \) and \( t \), we have:

\[
\frac{\partial \lambda}{\partial t} + \lambda \frac{\partial \lambda}{\partial x} = 0
\]  

(7.1)

since \( \lambda \) does not vary in the neighbourhood of a geometrical point travelling with velocity \( W \); this is in fact the definition of \( W \). Again if we imagine another geometrical point to travel with the "wave", we have:

\[
\frac{\partial \lambda}{\partial t} + \lambda \frac{\partial \lambda}{\partial x} = \lambda \frac{dV}{d\lambda} \frac{\partial \lambda}{\partial x} \]

(8.1)

the second member expresses the rate at which two consecutive wave - crests are separating from one another. Combinig 7.1 and 8.1 we are led to the formula 5.

This formula admits of a simple geometrical representation. If a curve be constructed with \( \lambda \) as abscissa and \( V \) as ordinate, the group - velocity will be represented by the intercept made by the tangent on the axis of \( V \).

Thus in Figure 1, \( PN \) represents the phase-velocity for the wave - length \( ON \), and \( OT \) represents the group-velocity. The frequency of vibration, it may be noticed, is represented by the tangent of the angle \( PON \).
5- Some Particular Cases

From equation 2, we have:

\[ \omega = \frac{d(mV)}{dm}, \quad \text{or} \]

\[ \omega = \frac{1}{\sqrt{1 + \frac{d \xi}{d \xi_m}}} \]  \hspace{1cm} (9.1)

Thus if

\[ \omega \propto \lambda^k \]

then

\[ \omega = (1 - \kappa) \lambda^k \]

The following particular case, which are taken from Rayleigh’s paper 14, are written down for future reference:

\begin{align*}
V \propto \lambda & \quad \Rightarrow \quad W = 0 \quad \text{Reynold’s disconnected pendulums} \\
V \propto \lambda^k & \quad \Rightarrow \quad W = \frac{1}{2} \xi \quad \text{Deep-water gravity waves} \\
V \propto \lambda^\xi & \quad \Rightarrow \quad W = \xi \quad \text{Capillary water-waves} \\
V \propto \lambda^\theta & \quad \Rightarrow \quad W = \nu \quad \text{Aerial waves, etc.} \\
V \propto \lambda^1 & \quad \Rightarrow \quad W = 2 \xi \quad \text{Flexural waves.}
\end{align*}

The Rate at Which Energy Is Transmitted by Waves

6- General Considerations

In this chapter we shall show how to calculate the energy of a progressive wave and that of a standing wave, and the rate at which energy is propagated by a progressive wave, and how this is related to our problem of group-velocity;

In the case of a progressive wave the wave front advances with a definite velocity but it does not follow that this is the rate of transmission of the energy, and there is no reason to suppose that they hang on the energy at the same rate as the wave-front advances. This question was discussed by Osborne Reynolds in a paper from which the following illustrations are borrowed, (Ref. 17):
If a number of small balls are suspended by threads so that the balls all hang in a row, the threads being of the same length; and if the balls be then set swinging in succession in planes perpendicular to the row, as by running the finger along them, the motion will present the appearance of a series of waves propagated from one end of the row to the other, but in reality each swings independently of its neighbour and there is no "communication of energy". If however the balls are connected by an elastic string and any one be given a transverse motion, it will communicate its motion to the others so that now there is a transmission of energy and the rate at which the first ball gives up energy to the others will clearly depend on the tension of the string.

As another illustration: - If a rope be laid out on the ground in a straight line with one end fixed and an upward jerk be given to the other end, a wringle will travel along the rope to the other end leaving the rope behind it straight and at rest on the ground. This is a case in which energy is transmitted at the same rate as the wave.

The majority of the cases with which we are interested, cases of water waves or atmospheric waves, are cases intermediate between the two just considered; energy is transmitted but at a rate less than the wave-velocity.

7- General Equation of Energy

The equations of motion in Eulerian form are:
\[ \begin{align*} 
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} &= X - \frac{\partial p}{\partial x} \\
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} &= Y - \frac{\partial p}{\partial y} \\
\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} &= Z - \frac{\partial p}{\partial z} 
\end{align*} \]

where \( u, v, w \) are the components of the velocity, along the \( x, y, z \) axes, respectively; \( X, Y, \) and \( Z \) are the components of the external forces acting on the fluid whose motion we are considering, \( p \) is the pressure; and \( \rho \) is the density.

The equation of continuity, in the same notation, is:

\[ \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} + \rho \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0 \tag{11.1} \]

Now, assuming that the extraneous forces have a potential \( \Omega \); we have:

\[ \begin{align*} 
X &= -\frac{\partial \Omega}{\partial x} \\
Y &= -\frac{\partial \Omega}{\partial y} \\
Z &= -\frac{\partial \Omega}{\partial z} 
\end{align*} \tag{13.1} \]

\( \Omega \) denotes the potential energy, per unit mass, at the point \((x, y, z)\), in respect of forces acting at a distance.

Assuming that the field of extraneous force is constant with time, i.e. \( \frac{\partial \Omega}{\partial t} = 0 \), we get from equation 11.1:
\[
\frac{df}{dt} (T + E) = - \iiint (u \frac{\partial p}{\partial x} + v \frac{\partial p}{\partial y} + w \frac{\partial p}{\partial z}) \, dx \, dy \, dz \quad \ldots \quad (14.1)
\]

where
\[
T = \frac{1}{2} \iiint \rho (u' + v' + w') \, dx \, dy \, dz \quad \ldots \quad (15.1)
\]

and
\[
E = \iiint \rho \omega \, dx \, dy \, dz \quad \ldots \quad (15.1)
\]

T and E denote the kinetic energy and the potential energy in relation to the field of extraneous force, of the fluid which at the moment occupies the region in question.

If \( ds \) denotes an element of the boundary surface of the fluid, and \( \xi, \eta, \mu \), and \( n \) denote the direction cosines of the inwardly directed normal to that element, then equation 14.1, in virtue of Green's theorem, becomes:

\[
\frac{df}{dt} (T + E) = \iiint (\xi u + \eta v + \mu w) \, p \, ds \quad \ldots \quad (16.1)
\]

The latter integral expresses the rate at which the pressure \( p \, ds \) exerted from without on the various elements of the boundary are doing work. Hence the total increase of energy, kinetic and potential, of any portion of the fluid, is equal to the work done by the pressures on the surface.

Now we assume the existence of a velocity-potential \( \psi \), so that:

\[
\begin{align*}
u &= - \frac{\partial \psi}{\partial x} \\
v &= - \frac{\partial \psi}{\partial y} \\
\omega &= - \frac{\partial \psi}{\partial z}
\end{align*}
\quad \ldots \quad (17.1)
\]

And since the motion is irrotational,
\[ \nabla^2 \varphi = 0 \]
so that
\[ \iiint \left[ (\frac{\partial \varphi}{\partial x})^2 + (\frac{\partial \varphi}{\partial y})^2 + (\frac{\partial \varphi}{\partial z})^2 \right] dx \, dy \, dz = -\iiint \varphi \frac{\partial^2 \varphi}{\partial n^2} \, dS \quad \ldots \quad (18.1) \]
therefore the kinetic energy is
\[ 2T = -\rho \iiint \varphi \frac{\partial \varphi}{\partial n} \, dS \quad \ldots \quad \ldots \quad \ldots \quad (19.1) \]

8- The Energy of Progressive Gravitational Waves

Considering a train of progressive waves at the surface of an incompressible fluid of depth \( h \), given by
\[ \eta = a \sin (mk - nt) \]
\[ \varphi = \frac{2a}{n} \frac{\cosh(m(1+h))}{\cosh mh} \cos (mk - nt) \]
if we calculate the energy of the fluid between two vertical planes parallel to the direction of propagation at unit distance apart, we have for a single wave - length:

the potential energy: \[ E = \frac{1}{2} \rho \int_0^L \eta^2 \, dx = \frac{1}{2} \rho \rho a^2 \lambda \]
the kinetic energy: \[ T = -\frac{1}{2} \rho \int \varphi \frac{\partial \varphi}{\partial n} \, dS = \frac{1}{2} \rho \rho a^2 \lambda \]

Hence the total energy per wave - length is
\[ T + E = \frac{1}{2} \rho \rho a^2 \lambda \]
and that it is half kinetic and half potential.

Also, considering any length in the fluid, in the direction of propagation, which is either an exact number of wave - lengths or is so large that the energy of a fractional part of a wave - length may be neglected in comparison with the energy of the whole, it follows that it is correct to say that "the energy of a progressive train of waves is half
kinetic and half potential".

9- The Energy of Stationary Gravitational Waves

For stationary waves:
\[ \eta = a \sin m \phi \cos n t \]

and
\[ \phi = \frac{2a}{m} \cos \left( \frac{3 + k}{m} \right) \sin m \phi \cos n t \]

The potential energy will therefore be
\[ E = \frac{1}{4} g \rho a^2 \lambda \cos^2 n t \]

and the kinetic energy
\[ T = \frac{1}{4} g \rho a^2 \lambda \]

Hence the total energy at any time is:
\[ T + E = \frac{1}{4} g \rho a^2 \lambda \]

and the amounts of kinetic and potential energy change continuously with time.

10- The Equation of Pressure

From the equations of motion (11.1), we get:
\[ \left( \frac{d \psi}{dt} - X \right) d \alpha + \left( \frac{d \psi}{dt} - Y \right) d \beta + \left( \frac{d \psi}{dt} - Z \right) d \gamma + \frac{1}{2} d \rho = 0 \]

And assuming, as before, the existence of a functional relation between the pressure and the density, i.e. assuming barotropic conditions, we get by integration:
\[ \int \frac{d \rho}{\rho} = \frac{\partial \phi}{\partial t} + \frac{1}{2} \phi^2 + \Omega = F(t) \]

where \( F(t) \) is in general an arbitrary function of the time.

It is possible, however, to regard this function of time as being contained in the term \( \frac{\partial \phi}{\partial t} \). In 21.1 \( q \) denotes the velocity.

If the fluid be homogeneous and incompressible, the last equation, 21.1, becomes:
\[ \frac{\psi}{t} - \frac{\partial \phi}{\partial t} + \frac{1}{2} q^2 + \Omega = F(t) \]

\[ \frac{\psi}{t} - \frac{\partial \phi}{\partial t} + \frac{1}{2} q^2 + \Omega = F(t) \]
Rate of Transmission of Energy in Regular Waves

The case of regular waves was discussed by Lord Rayleigh in his paper "On Progressive Waves" (Ref.16), which is summarised below:

In the case of regular waves the ratio of the energy propagated to that of the passing waves is

\[ \frac{w}{v} \]

Accordingly:

The energy propagated in unit time: The energy contained (on an average) in unit length = \[ \frac{\rho \frac{dv}{dt}}{\rho} \], by ...(2.1)

As an example, consider the case of small irrotational waves in water of finite depth \( h \). If \( z \) be measured downwards from the surface and the elevation \( \eta \) of the wave be

\[ \eta = a \cos (\omega t - m x) \]

the corresponding velocity - potential \( \psi \) is:

\[ \psi = -V a \frac{\cosh m(\frac{3}{2} - h)}{\sinh m h} \sin (\omega t - m x) \]

The velocity of progress is

\[ v = \frac{\partial \psi}{\partial x} = \frac{\frac{3}{2} a m \cosh m h}{\sinh m h} \]

We now calculate the energy contained in a length \( x \), which is supposed to include so great a number of waves that fractional parts may be left out of account.

The potential energy is:

\[ E = \frac{1}{2} \rho \int \int \int \frac{\partial \psi}{\partial x} \frac{\partial \psi}{\partial x} dx \, dy \, dz = \frac{\rho}{2} \int \int \int \left( \psi \frac{\partial \psi}{\partial x} \right) dx \, dy \, dz = \frac{1}{2} \rho a^2 \frac{h}{6} \]

The kinetic energy is

\[ K = \frac{1}{2} \rho \int \int \left[ \left( \frac{\partial \psi}{\partial y} \right)^2 + \left( \frac{\partial \psi}{\partial z} \right)^2 \right] dx \, dy \, dz = \frac{1}{2} \rho \int \int \left( \frac{\partial \psi}{\partial y} \right)^2 \, dx \, dy \, dz = \frac{1}{2} \rho a^2 \frac{h}{6} \]

Therefore the total energy is
\[ T + E = \frac{1}{2} \rho a^4 x \] (25.1)

Now, the rate of transmission of energy is measured by taking a plane intersecting the axis of the wave and at right angle to the direction of propagation, and determining the work \( W_c \) that must be done in order to sustain the motion of that plane in the face of the fluid pressures.

The variable part of the pressure, \( \delta p \), at depth \( z \) is given by:
\[ \delta p = -\rho \frac{\partial \phi}{\partial x} = -mV a \frac{\sinh (3h - 4x)}{\sinh mh} \cos (nt - mx) \]

and the horizontal velocity is
\[ u = \frac{\partial \phi}{\partial x} = mV a \frac{\sinh (3h - 4x)}{\sinh mh} \cos (nt - mx) \]

so that the work is
\[ W_c = \int \int \delta p \frac{\partial \phi}{\partial x} dx dy dt = \frac{1}{2} \delta \rho a^2 V t \left[ 1 + \frac{\delta mh}{\sinh 2mh} \right] \]

From the value of \( V \), we get:
\[ \frac{d(mV)}{dn} = \frac{1}{2} V \left[ 1 + \frac{1}{V^2} \frac{d(mV)}{dn} \right] \equiv \frac{1}{2} V \left[ 1 + \frac{\delta mh}{\sinh 2mh} \right] \]

It is thus verified that the work for a unit time is
\[ W_c = \frac{d(mV)}{dn} \times \text{energy in unit length,} \]

or, what amounts to the same thing:

"The energy is transmitted at a rate equal to the group-velocity".

In the same paper Lord Rayleigh proves further that this result is not only true for the special case considered above, but it is true also for all cases of regular
progressive waves. His method of proof is, briefly, to assume a resisting force acting on the particles of the medium, and to equate the energy dissipated by this force to the energy transmitted by the wave making use of the assumption that the "kinetic energy is equal to the potential energy", which is true as long as we are referring either to integral number of wave-lengths, or to a space so considerable that fractional parts of waves may be left out of account.
CHAPTER II

GROUP - VELOCITY OF WAVES IN A
NON - VISCOSUS HOMOGENEOUS FLUID ON A NON -
ROTATING EARTH

12- In this chapter some relatively simple cases of wave propagation will be treated. The fluid may be assumed incompressible and homogeneous, and the waves may be plane, i.e having straight wave-fronts. The sphericity and rotation of the earth may be neglected.

13- Surface Waves

We shall first consider waves on an unlimited sheet of fluid under no force but gravity. The axis of x is taken in the undisturbed surface in the direction of propagation of the waves, and the axis of z vertically upwards. The motion is irrotational, and therefore it has a velocity potential $\psi$ which satisfies the equation:

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \quad \ldots \ldots \ldots \quad (1.2)$$

throughout the liquid, and

$$\frac{\partial \psi}{\partial x} = 0 \quad \ldots \ldots \ldots \ldots \ldots \quad (2.2)$$

at the rigid boundary.

The pressure is given by:

$$\rho = \frac{\partial \psi}{\partial x} - g \frac{\partial^2 \psi}{\partial y^2} + f(x) \quad \ldots \ldots \quad (3.2)$$
The free surface is a surface of equipressure, i.e.

\[ p = \text{const.} \]

therefore:

\[ \frac{\partial p}{\partial t} + u \frac{\partial p}{\partial x} + \omega \frac{\partial p}{\partial y} = 0 \]

or:

\[ \frac{\partial p}{\partial t} - \frac{\partial \varphi}{\partial x} \frac{\partial p}{\partial x} - \frac{\partial \varphi}{\partial y} \frac{\partial p}{\partial y} = 0 \]

(4.2)

at the surface.

Neglecting quantities of second order of magnitude, and assuming the function \( F(t) \) to be included in \( \frac{\partial \varphi}{\partial z} \), we get:

\[ \frac{\partial \varphi}{\partial t} + g \frac{\partial \varphi}{\partial z} = 0 \]

(5.2)

This condition holds at the surface.

If \( \eta \) denotes the elevation of the free surface at time \( t \) above the point whose abscissa is \( x \), the equation of the surface will be of the form:

\[ \eta = f(x,t) = 0 \]

where \( f(x,t) \) is a function of \( x \) and \( t \) only. And this, being a boundary, must satisfy the boundary condition:

\[ \frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} + v \frac{\partial f}{\partial y} + \omega \frac{\partial f}{\partial z} = 0 \]

Hence:

\[ \frac{\partial f}{\partial t} + u \frac{\partial f}{\partial x} - \omega = 0 \]

But; \( \frac{\partial f}{\partial t} = \dot{\eta} \), and \( \frac{\partial f}{\partial x} = \frac{\partial \eta}{\partial x} \) is the tangent of the slope of the free surface which is of second order of magnitude and could be neglected, therefore; so that the equation becomes:

\[ \dot{\eta} = \omega = - \frac{\partial \varphi}{\partial z} \]

(6.4)
Let us now apply these equations to the case of a homogeneous fluid of uniform depth \( h \), either of unlimited extent or contained in a contained in a canal with parallel vertical sides at right angles to the wave fronts.

If simple harmonic waves are propagated, then the solution of the equations will have the following form:

\[ \varphi = C(z) e^{i(mx - nt)} \]

Substituting in 1.2, we get:

\[ \frac{\partial^2 C(z)}{\partial z^2} - m^2 C(z) = 0 \]

\[ C(z) = A e^{mz} + B e^{-mz} \]

and:

\[ \varphi = (A e^{mz} + B e^{-mz}) e^{i(mx - nt)} \]

From the boundary condition: \( \frac{\partial \varphi}{\partial z} = 0 \), we get:

\[ \frac{\partial \varphi}{\partial z} = 0 \quad \text{when} \quad z = -h \]

\[ A e^{-mh} = B e^{mh} = \frac{1}{2} K, \text{ say} \]

So that:

\[ \varphi = K \cosh mz \cos(mx - nt) \quad \ldots \quad (7.2) \]

Substituting this value in the surface condition 5.2, and taking \( z = 0 \), we get:

\[ n^2 = q m \tanh mh \quad \ldots \quad (7.3) \]

And since the velocity of propagation is:

\[ V = \frac{m}{n} \]

and the wave-length is:

\[ \lambda = \frac{2n}{m} \]

it follows that
\[ V^2 = \frac{g}{m} \tanh \frac{nh}{\lambda} = \frac{g \lambda}{m} \tanh \frac{2nh}{\lambda} \]  

It can be easily shown that:

\[ K = \frac{m}{m_R} \frac{a}{\sinh nh} \]

where \( a \) is the amplitude.

hence:

\[ \varphi = \frac{m}{m_R} \frac{N_m \eta (\delta + h)}{\sinh nh} \cos (m x - nt) \]

or,

\[ \varphi = \frac{g a}{m} \frac{N_m (3h)}{\sinh nh} \cos (m x - nt) \]

If the fluid is deep, i.e., \( \frac{2nh}{\lambda} \) is large, we get

\[ \tanh \frac{2nh}{\lambda} \rightarrow 1 \]

\[ V^2 = \frac{g \lambda}{2m} \]

And if the fluid is shallow, i.e., \( \frac{2nh}{\lambda} \) is small, we get

\[ \tanh \frac{2nh}{\lambda} \rightarrow \frac{2nh}{\lambda} \]

\[ V^2 = gh \]

15- The Group - velocity

It is clear that equation 9.2 gives the phase - velocity of a single sinuous wave train. To get the group - velocity of such a wave motion equation 5.1 may be used. Thus we get:

\[ W = \frac{1}{2} V \left( 1 + 2 m \lambda \coth 2 m h \right) \]

Hence the ratio of the group - velocity to the phase - velocity in the case of a fluid of depth \( h \), is:

\[ \frac{W}{V} = \frac{1}{2} + \frac{m h}{\sinh 2m h} \]
When \( h \) is large, i.e., in the case of deep fluid, \( \sinh \frac{2\pi h}{l} \) is large also, and the ratio \( \frac{m^2 h}{2} \frac{\sinh \frac{2\pi h}{l}}{1 + \frac{2\pi h}{l}} \) tends to zero as \( h \) increases and tends to infinity; hence, in that case

\[
W = \frac{1}{2} V = \frac{1}{2} \sqrt{\frac{g \lambda}{2 \rho}}
\]

The same result could be found by substituting equation 10.2 in equation 5.1 directly.

On the other hand, when \( h \) is small compared with the wave-length \( \lambda \), \( \sinh \frac{2\pi h}{l} \) approaches \( 2\pi h \), and

\[
W = V = \sqrt{g h}
\]

which means that the group-velocity is equal to the phase-velocity in this case. This result is self-evident, since the phase-velocity is not a function of wave-length, and the medium does not act as a dispersive medium to such waves.

16- Waves at The Common Surface Between Two Fluids

Suppose a fluid of density \( \rho' \) to be moving with velocity \( U' \) over another fluid of density \( \rho \) moving in the same direction with velocity \( U \); and assume both fluids to be unlimited in depth.

Let the velocity-potentials of the disturbed motion be \( \Phi \) and \( \Phi' \), respectively; and the perturbation velocity-potentials be \( \Phi' \) and \( \Phi \), and assume that the perturbation quantities are small compared with the disturbed quantities. Taking the \( x \)-axis in the direction in which the fluids are moving and in the undisturbed common surface, and the \( z \)-axis vertically upwards, we may write:
$$\overline{\varphi} = -\mathbf{U}_x + \varphi', \text{ for the upper fluid},$$

and $$\overline{\varphi} = -\mathbf{U}_x + \varphi, \text{ for the lower fluid}.$$

Hence if $\eta$ be the ordinate of the displaced surface, and substituting

$$F = \gamma - \eta$$

in the boundary condition

$$\frac{\partial F}{\partial t} + \overline{\mathbf{u}} \cdot \frac{\partial F}{\partial x} + \mathbf{v} \cdot \frac{\partial F}{\partial y} + \mathbf{w} \cdot \frac{\partial F}{\partial z} = \sigma \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (11.2)$$

we get, after neglecting terms of second order:

$$\left\{ \begin{align*}
\frac{\partial \eta}{\partial t} + \mathbf{v} \cdot \frac{\partial \eta}{\partial x} & = -\frac{\partial \varphi}{\partial y} \\
\mathbf{v} \cdot \frac{\partial \eta}{\partial x} & = -\frac{\partial \varphi}{\partial y}
\end{align*} \right\} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (12.2)$$

These are the kinematic conditions to be satisfied for $z = 0$.

The formula for the pressure in the lower fluid is:

$$p = \frac{\partial \varphi}{\partial t} + \frac{1}{2} \left[ \mathbf{v} \cdot \left( \frac{\partial \varphi}{\partial x} \right)^2 + \left( \frac{\partial \varphi}{\partial y} \right)^2 \right] - g \gamma$$

$$= \frac{\partial \varphi}{\partial t} + \mathbf{v} \cdot \frac{\partial \varphi}{\partial x} - g \gamma,$$

after omitting terms of second order of magnitude.

And a similar formula may be derived for the upper fluid.

Since at the common surface $p = p'$, we get:

$$\rho \left( \frac{\partial \varphi'}{\partial t} + \mathbf{v} \cdot \frac{\partial \varphi'}{\partial x} - g \eta \right) = \rho' \left( \frac{\partial \varphi'}{\partial t} + \mathbf{v} \cdot \frac{\partial \varphi'}{\partial x} - g \eta \right)$$

The solution may be assumed to be of the form:
\[ \varphi = C e^{m \varphi + i(nt-mx)} \]
\[ \varphi' = C' e^{-m \varphi + i(nt-mx)} \]
\[ \eta = \alpha e^{i(nt-mx)} \]

Substituting in 12.2, we get:
\[ i(m \varphi - u) \alpha = -mc \]
\[ i(m \varphi' - u') \alpha = mc' \]

And from the equality of pressure, we get:
\[ \rho \left[ i(m \varphi - u) \alpha - ga \right] = \rho' \left[ i(m \varphi' - u') \alpha - ga \right] \]

Hence
\[ \rho(m \varphi - u) + \rho'(m \varphi' - u') = g \rho (\rho - \rho') \]

or
\[ V = \frac{n}{m} = \frac{\rho \varphi + \rho' \varphi'}{\rho + \rho'} - \frac{\delta^2}{(\rho + \rho')^2} \left( n - \varphi \right)^2 \]

or
\[ V = \alpha + (\frac{\delta}{\rho} \beta - \gamma \theta)^2 = \alpha + \delta \]

Where
\[ \alpha = \frac{\rho \varphi + \rho' \varphi'}{\rho + \rho'} \]
\[ \beta = \frac{\rho - \rho'}{\rho + \rho'} \]
\[ \gamma = \frac{\rho \rho'}{(\rho + \rho')^2}, \text{ and} \]
\[ \delta = \varphi - \varphi' \]

17- Next, suppose that the two fluids of the last problem are confined between two fixed horizontal planes, such that the depth of the upper fluid is \( h' \) and that of the lower \( h \).

Let us make the motion steady by superposing on
the whole mass the velocity \( \mathbf{V} \), thereby bringing the wave
form to rest in space.

Let the velocity - potential at the lower layer
be:
\[
\Phi = -(\mathbf{U} - \mathbf{V}) x + A \cosh (z + b) \cos \mu x \quad (15.2a)
\]
and at the upper layer be:
\[
\Phi' = -(\mathbf{U}' - \mathbf{V}) x + A' \cosh (z + b') \cos \mu x \quad (15.2b)
\]
and let the stream function at the lower layer be:
\[
\psi = -(\mathbf{U} - \mathbf{V}) y - A \sinh (z + b) \sin \mu x \quad (16.2a)
\]
and at the upper layer be:
\[
\psi' = -(\mathbf{U}' - \mathbf{V}) y - A' \sinh (z + b') \sin \mu x \quad (16.2b)
\]
The displacement of the common surface is given
by:
\[
\eta = a \sin \mu x .
\]
The boundary condition at the common surface is \( \psi = \psi' \), which
gives:
\[
-(\mathbf{U} - \mathbf{V}) a - A \sinh mb = 0
\]
\[
-(\mathbf{U}' - \mathbf{V}) a + A' \sinh mb' = 0
\]
neglecting second order quantities.

The expressions for the pressures are:
\[
\frac{p}{\rho} + g \beta + \frac{1}{2} \left[ \left( \frac{\partial \Phi}{\partial x} \right)^2 + \left( \frac{\partial \Phi}{\partial y} \right)^2 \right] = \text{const.}, \quad \text{and}
\]
\[
\frac{p'}{\rho'} + g \beta' + \frac{1}{2} \left[ \left( \frac{\partial \Phi'}{\partial x} \right)^2 + \left( \frac{\partial \Phi'}{\partial y} \right)^2 \right] = \text{const.}
\]
And since at the common surface \( p = p' \), we get:
\[
9 (\rho - \rho') = (\mathbf{U} - \mathbf{V}) \rho \coth mb + (\mathbf{U}' - \mathbf{V}') \rho' \coth mb'
\]
or:
\[
\mathbf{V} = \frac{\mathbf{U} \rho \coth mb + \mathbf{U}' \rho' \coth mb'}{\rho \coth mb + \rho' \coth mb'}
\]
\[
+ \sqrt{\left( \frac{\mathbf{U} \rho \coth mb + \mathbf{U}' \rho' \coth mb'}{\rho \coth mb + \rho' \coth mb'} \right)^2 - \frac{\mathbf{U} \rho \coth mb + \mathbf{U}' \rho' \coth mb'}{\rho \coth mb + \rho' \coth mb'}} \quad (17.2)
\]
If the upper liquid is unlimited in depth, i.e. $h' \to \infty$, $c_{th} \to h' + 1$, and the last formula reduces to:

$$g^{(r-r')} = (V-V')^{m \rho} c_{th}^{2} \left[ (V-V')^{m \rho} \right]^{\cdots} \text{(18.2)}$$

18- The Group - Velocity

From the third of equations 13.2, we have:

$$\eta = \frac{a}{(t - m x)} \text{, or} \eta = a c_{th} (t - m x) + i a \sin (nt - m x)$$

which shows that the individual waves have a sinuous form, and therefore equation 4.1 is applicable to this case also.

Now, equation 4.1 is:

$$W = \frac{v + m}{dV} \left[ \frac{dV}{dt} \right] \text{ (4.1)}$$

and equation 14.2 could be written in the form:

$$V = \alpha \sqrt{\frac{\beta}{m} - \nu} \text{ (19.2)}$$

where:

$$\alpha = \frac{\rho V + \rho'}{\rho + \rho'} \text{ (a)}$$

$$\beta = \frac{\rho' - \rho}{\rho' + \rho} \text{ (2.0.2)}$$

and

$$r = \frac{\rho \rho'}{(\rho + \rho')^{2}} \left( V - V' \right)^{2} \text{ (c)}$$

The double sign simply means that the waves can travel in the positive- or in the negative- direction of the x-axis.

Taking the positive sign and substituting in 4.1, we get:

$$W = V - \frac{\beta}{2m} \frac{1}{V - \alpha} \text{ (21.2)}$$
In equation 19.2, since the radical is always positive,

\[ V > \alpha \]

And in 21.2, since

\[ V - \alpha > \sigma \]

and \( \beta > \sigma \), if the stratification is stable, therefore:

\[ W < V \]

Hence in the case we are considering, the group-velocity is always less than the phase-velocity; a result which might have been anticipated from the discussion given in Art. 4, and from the table given in Art. 5.

19- Billow Clouds

The theory of the billow clouds was discussed by Haurwitz in his book "Dynamic Meteorology" (Ref. 24), where he considers the case of zero phase-velocity. Let us investigate here the case of stationary groups of waves, which is more likely associated with billow clouds.

When the group is stationary, we have:

\[ W = 0 \]

down therefore, equation 21.2 becomes:

\[ \frac{V - \sigma}{\beta} \frac{1}{V - \alpha} = 0 \]

where \( V \) is the phase-velocity for a group of stationary waves.

Solving the last equation for \( V \), we get:

\[ V = \frac{1}{2} \alpha \pm \frac{1}{2} \sqrt{\alpha^2 + \frac{2\beta}{\nu}} \quad (21.2) \]
Since \( b > 0 \)

therefore \( \frac{1}{2} \sqrt{\alpha^2 + \frac{c^2 a^2}{a^2}} \geq \frac{1}{2} \alpha \)

And if \( U, U' > 0; \alpha > 0 \)

Taking \( \rho = \rho' \), we get:

\[ \alpha \approx \frac{1}{2} (U + U') \]

and considering the positive sign of 22.2, we get:

\[ V_t \geq \frac{1}{2} (U + U') \]

This shows that, in the case of billow cloud formation, the individual waves travel with a velocity greater than the mean velocity currents above and below the common surface; which result is in variance with the usual assumption made by former writers. For former writers have assumed that billow clouds move with a velocity equal to the mean velocity of the two currents.

Taking the negative sign of equation 22.2, we get:

\[ V_t < 0 \]

That is: the individual waves may travel in the negative direction with velocities which may vary within wide limits depending upon the absolute value of the radical. It is seen that in this case, the value of \( V_t \) decreases as \( b \) decreases, i.e. as \( \rho - \rho' \) decreases, and when

\[ \rho = \rho', \hspace{1cm} a = d \]

\[ b = 0, \hspace{1cm} V_t = 0 \]

which means that in this case the individual waves will also be
stationary. This result is almost self-evident; since from equation 4.1, if \( V = 0 \), \( W = 0 \).

Considering equation 4.2, we see that in the case when \( \rho \neq \rho' \), the velocity of the individual waves, i.e. the phase-velocity, is independent of the wave-length. The medium, therefore, is no more dispersive; and, hence, the group-velocity is equal to the phase-velocity.

Let us go back to the equation of billow clouds

\[
\sqrt{\gamma} - \frac{\frac{\gamma}{\alpha}}{2m} < \frac{1}{\sqrt{\gamma - \alpha}} = 0
\]

and compute the wave-length of the group of waves in which the clouds are formed.

Substituting the value of \( V \), from 4.2, and taking the positive sign of the radical, we get:

\[
\alpha + \sqrt{\frac{\gamma}{\alpha} - \rho} = \frac{\gamma}{\alpha} - \rho = 0 \quad \text{or}
\]

\[
4 \rho (\alpha + \rho) - 4 \gamma \beta (\alpha + \rho) \mu + \gamma \beta \mu = 0
\]

Solving for \( \mu \), we get:

\[
\mu = \frac{\gamma\beta}{2\rho} \left[ 1 \pm \sqrt{1 - \frac{\rho}{\alpha + \rho}} \right]
\]

and since \( m = \frac{\mu}{\alpha} \), therefore:

\[
\lambda = \frac{4 \gamma \beta}{\alpha \left[ 1 + \sqrt{1 - \frac{\rho}{\alpha + \rho}} \right]}
\]

(23.1)

The special case of \( U = U' \) could be solved directly from the equation

\[
4 \rho (\alpha + \rho) - 4 \gamma \beta (\alpha + \rho) \mu + \gamma \beta \mu = 0
\]

by setting \( \rho = 0 \) and solving for \( \mu \). Thus we get:

\[
\mu = \frac{\gamma\beta}{4\alpha^2} \quad \text{or}
\]
\[ \lambda = \frac{8 \pi \alpha e^2}{\sqrt{\beta}} \tag{24.2} \]

Introducing temperature instead of density, and since

\[ \rho = \frac{p}{RT} \quad \text{and} \quad \rho' = \frac{p'}{RT'} \quad \text{and since} \quad \rho = \rho' \]

equations 14.2 become:

\[ \alpha = \frac{7'V + 7'V'}{7' + 7'} \]

\[ \beta = \frac{7' - 7}{7' + 7'} \]

\[ = \frac{\Delta 7}{7' + 7'} \quad \text{and} \]

\[ \nu = \frac{7'7''}{(7' + 7')^2} (V' - V')^2 \]

\[ = \frac{7'7'}{(7' + 7')^2} (\Delta V) \]

The following table gives the wave-lengths of billow groups, corresponding to the wave-lengths given in Haurwitz's book.

<table>
<thead>
<tr>
<th>T (deg C)</th>
<th>( \Delta U ) m/sec</th>
<th>( \lambda ) cloud m</th>
<th>( \lambda ) group m</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.6</td>
<td>14.8</td>
<td>6160</td>
<td>148000</td>
</tr>
<tr>
<td>1.0</td>
<td>4.0</td>
<td>1165</td>
<td>210000</td>
</tr>
<tr>
<td>4.8</td>
<td>6.7</td>
<td>1060</td>
<td>460000</td>
</tr>
<tr>
<td>6.6</td>
<td>6.0</td>
<td>( \infty )</td>
<td>( \infty )</td>
</tr>
</tbody>
</table>

In computing this table the following values were assumed: \( T = 270 \) C, and \( U' = 10 \) m/sec.

Equation 23.2 gives two values for the wave-length of the cloud, one corresponding to each sign.
Taking the positive sign and computing the wave-lengths for the cases given in Haurwitz' book (Ref. p 288), we get values of the same order of magnitude as those given by Haurwitz; but taking the negative sign, we get values greater than these by a factor of 10-100, as seen from the table given above. The positive sign of the radical makes \( \lambda = 0 \) when \( r = 0 \), and therefore it does not represent the length of the group of waves, it can only be taken to represent the length of the individual clouds. This is in agreement with our ideas of the group-velocity. Since we should expect some long waves corresponding to the group as a whole, over which are superposed the shorter wave-lengths of the individual waves. Hence the general appearance of the clouds should be a more or less continuous deck of clouds extending over a wide area, in which individual billow clouds can be recognised as regions of thicker clouds separated by regions of thinner clouds through which blue sky may or may not be visible.

The whole deck of clouds which forms in this way may be called the "billow-group", in distinction from the individual billow clouds. The billow-groups should be separated from each other by comparatively wide areas of more or less blue skies, in accordance with our ideas that the groups of waves are separated from each other by regions of almost undisturbed medium.

It may be noted here that formula 22.2 shows that the individual billow clouds should have a small velocity relative to the billow-group, as already explained in Art.19,
a velocity which may be in the direction of the winds or in the opposite direction. The individual billow clouds will then appear to form at one limit of the billow - group and travel slowly in the group developing as they approach the center and dissipating gradually as they approach the other limit at which they dissipate completely.

20- We might go back to the case of two fluids having a lower rigid boundary which was discussed in Art. 17, and investigate the group - velocity.

It is readily seen that equation 5.1 is applicable here also, and that the procedure followed in the last article may well be repeated, starting with formula 17.2. However this case is of no special interest, since most of the cases which interest the meteorologist may either be of the form treated in the last article or they may be of the case in which the lower fluid could be treated as shallow in comparison with the wave-length. Groups of waves at the surface of a low inversion may be an example of this. However in the case of a shallow lower layer, the group - velocity is equal to the phase-velocity, and the classical methods of treatment become identical with the methd followed here.

We therefore refrain from treating any of these cases, and instead we will try to investigate the energy propagation associated with such waves, and find the relation between the rate at which energy is propagated and the group- velocity, and the implications of this on the weather condition assuming that the original assumptions apply to the atmosphere
21. The Energy of a Progressive Wave at The Common Surface Between Two Fluids

Let us consider the case discussed in Art. 16, in which the two fluids are assumed unlimited in depth.

From Art. 8, we have for the potential and kinetic energies:

\[ E = \frac{1}{2} g \rho \int_0^\lambda \eta \, dx \quad \ldots \quad (26.2) \]

and

\[ T = \frac{1}{2} \rho \int_0^\lambda \left( \eta \frac{\partial \eta}{\partial \xi} \right) \, dx \quad \ldots \quad (27.2) \]

Taking the real parts of equations 13.2, we have:

\[ \phi = -\nu \xi + \xi \frac{m^2}{c_0} (nt - m\xi) \quad \ldots \quad (28.2) \]

\[ \phi' = -\nu' \xi + \xi' \frac{m^2}{c_0} (nt - m\xi) \quad \ldots \quad (29.2) \]

and

\[ \eta = \alpha \phi (nt - m\xi) \quad \ldots \quad (30.2) \]

Considering the lower layer first, we get for the potential energy:

\[ E_i = \frac{1}{2} g \rho \int_0^\lambda \phi' (nt - m\xi) \, dx \]

or

\[ E_i = \frac{1}{4} g \rho \phi \lambda \]

and for the kinetic energy:

\[ T_i = \frac{1}{2} \rho c^2 m \int_0^\lambda \phi' (nt - m\xi) \, dx \]

or

\[ T_i = \frac{1}{4} \rho c^2 m \lambda \]

and substituting the relation

\[ c = -\eta (nt - m\xi) \]

which we get from Art. 16, we get:

\[ T_i = \frac{1}{4} \rho \frac{\Delta a}{m} \lambda^2 (nt - m\xi)^2 \]
Following a similar procedure, we get for the upper layer:

\[ \xi = \frac{1}{4} \mathbf{g} a^3 \lambda, \text{ and} \]

\[ T = \frac{1}{4} \rho' \frac{m}{m'} a' (m-m') \]

Therefore the total potential energy and kinetic energy are:

\[ E = \frac{1}{4} \int \mathbf{g} a^3 \lambda (\rho' - \rho') \quad \ldots \ldots \quad (31.1) \]

and:

\[ T = \frac{1}{4} m \int \left[ (m-m') \frac{1}{\rho} + (m-m') (\rho')^2 \right] \]

Also from Art. 16, we have:

\[ (m-m') \frac{1}{\rho} + (m-m') (\rho')^2 = m m' (\rho' - \rho') \]

therefore:

\[ T = \frac{1}{4} \mathbf{g} a^3 \lambda (\rho' - \rho') \quad \ldots \ldots \quad (32.2) \]

From (31.1) and (32.2), we get:

Total energy = \[ E + T = \frac{1}{4} \mathbf{g} a^3 \lambda (\rho' - \rho') \quad \ldots \ldots \quad (33.2) \]

which is half potential and half kinetic as might have been anticipated from the discussion of article 8.

22-

To get the rate at which the energy is transmitted, we do not need to go through the whole procedure followed in Art. 11, since it follows directly from the discussion of that article and from the last article that the energy is being transmitted with the same velocity as that of the group of waves associated with it.

This shows that the centers of the groups of waves will also be the centers of activity. And in the case of billow cloud formations, it means that the centers of the groups will also be the centers of maximum precipitation and turbulence. This theory explains the periodicity of showers.
and the periodic ceiling fluctuations, as pointed out by Jacobs (Ref. 24), except that according to the theory of group-velocity the periods of such phenomena may be much longer; this can easily be seen from table I by comparing the wave-lengths of the billow clouds to those of the billow grous. The periods may have the same ratios.
CHAPTER III

GROUP VELOCITY OF WAVES ON
A ROTATING EARTH

24- The Equations of Motion on a Rotating Earth

On a rotating earth, when the only extaneous forces influencing the motion of the fluid are those of the pressure gradient, the simplified equations of motion, in their Eulerian form, become (Ref. 24):

\[
\begin{align*}
\frac{2u}{\partial t} + u \frac{2u}{\partial x} + v \frac{2u}{\partial y} + \omega \frac{2u}{\partial z} - 2\omega (u \sin \varphi - u \cos \varphi) &= - \frac{1}{\rho} \frac{\partial p}{\partial x} \\
\frac{2v}{\partial t} + u \frac{2v}{\partial x} + v \frac{2v}{\partial y} + \omega \frac{2v}{\partial z} + 2\omega \sin \varphi \cdot u &= - \frac{1}{\rho} \frac{\partial p}{\partial y} \\
\frac{2\omega}{\partial t} + u \frac{2\omega}{\partial x} + v \frac{2\omega}{\partial y} + \omega \frac{2\omega}{\partial z} - 2\omega \cos \varphi \cdot u &= - \frac{1}{\rho} \frac{\partial p}{\partial z} - g
\end{align*}
\]

where \( \omega \) is the angular velocity of the earth's rotation, and \( \varphi \) is the latitude.

And the equation of continuity is, as before,

\[
\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial z} = 0 \quad \quad (2.3)
\]

In this chapter the more general case of waves in a fluid on a rotating earth will be treated. We will start by considering long gravitational waves; assuming, as before, that the fluids under consideration are homogeneous and incompressible.
Long Gravitational Waves on an Unlimited Rotating Disc

We shall first consider waves in a fluid layer of even depth, rotating around a vertical axis. The lower boudary may be rigid, and the upper surface may be free. Such a case may be thought of as similar to motion at the pole, but it could well be applied to any place if the dimensions of the area under consideration are not so large as to be affected by the sphericity of the earth considerably.

The following solution was given by Sverdrup, (Ref. 23), but the general solution for a rotating fluid was first given by Lord Kelvin, (Ref. 11).

Let the z-axis coincide with the axis of rotation, and the x- and y-axes be supposed to rotate in their plane with the angular-velocity of the disc, viz. $\omega x$. The velocities of the fluid particles parallel to these axes are:

$$u = \omega y, \quad v = \omega x, \quad \text{and} \quad w$$

The ordinate of the free surface, in the undisturbed case, when it is in equilibrium under the influence of the gravitational force and the centrifugal force, may be called $z$. We then have:

$$z_0 = \frac{1}{2} \omega^2 r (x^2 + y^2) + c \cos \phi \quad (3.3)$$

It shall be supposed that the inclination of this surface is small, which means that $\frac{\omega^2 r}{g}$ is small, where

$$r = \sqrt{x^2 + y^2}$$

If the ordinate of the disturbed surface is called $z_0 + \eta$, the pressure in any point $(x, y, z)$ is given by
\[ P - \rho g = \frac{g}{\rho} (\eta + \eta^2 - \eta) \quad (4.3) \]

where \( \rho \) is the constant pressure at the surface.

From (4.3), we get:

\[ \frac{\partial \rho}{\partial x} = \frac{g}{\rho} \frac{\partial \eta}{\partial x}, \quad \text{and} \quad \frac{\partial \rho}{\partial y} = \frac{g}{\rho} \frac{\partial \eta}{\partial y} \quad \ldots \ldots \ldots \quad (5.3) \]

Now the equations of motion (1.3), for the case of a disc, could be written:

\[ \frac{d\eta}{dt} - \eta = - \frac{g}{\rho} \frac{\partial \rho}{\partial x}, \quad \ldots \ldots \ldots \ldots \quad (6.3) \]

where \( \frac{d\eta}{dt} \) has its usual meaning, \( \lambda = 2 \omega \sin \varphi \), and \( \omega \sin \varphi = \omega^* \) corresponds to the angular velocity of rotation of the earth.

For an incompressible fluid, like the one we are considering here, the equation of continuity becomes:

\[ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \ldots \ldots \ldots \ldots \quad (7.3) \]

Substituting from 5.3 in 6.3, we get:

\[ \frac{du}{dt} - \lambda = - \frac{g}{\rho} \frac{\partial \eta}{\partial x} \]

\[ \frac{dv}{dt} + \lambda = - \frac{g}{\rho} \frac{\partial \eta}{\partial y} \]

And neglecting quantities of second order, these become:

\[ \frac{du}{dt} - \lambda = - \frac{g}{\rho} \frac{\partial \eta}{\partial x}, \quad \frac{dv}{dt} + \lambda = - \frac{g}{\rho} \frac{\partial \eta}{\partial y}, \quad \ldots \ldots \ldots \ldots \quad (8.3) \]

And from the equation of continuity we get:

\[ \omega = - \int \left( \frac{\partial u}{\partial x} + \frac{\partial w}{\partial y} \right) \, dz, \quad \text{or} \]

\[ \omega = \frac{\partial \eta}{\partial t} = - h \frac{\partial u}{\partial x} - h \frac{\partial v}{\partial y} \quad \ldots \ldots \ldots \quad (9.3) \]
where $h$ denotes the depth.

On an unlimited rotating disc the equations 8.3 and 9.3 are satisfied by:

$$\eta = a \sin(nt - mx)$$

$$\frac{\partial \nu}{\partial y} = 0$$  \hspace{1cm} (10.3)

Substituting in 8.3 and 9.3, we get:

$$\frac{\partial u}{\partial t} - \nu = g a \cos(nt - mx)$$

$$\frac{\partial \nu}{\partial t} + \nu u = 0$$  \hspace{1cm} (11.3)

and

$$\frac{\partial \eta}{\partial t} = -h \frac{\partial u}{\partial x}$$  \hspace{1cm} (12.3)

From 10.3, we get:

$$u = \frac{g}{\nu} a \frac{m^2}{m^2 - \ell^2} \sin(nt - mx)$$

$$\nu = \frac{g}{u} a \frac{m^2}{m^2 - \ell^2} \cos(nt - mx)$$  \hspace{1cm} (13.3)

and from 11.3, we get:

$$\eta = \frac{g h}{\nu} \frac{m^2}{m^2 - \ell^2} a \sin(nt - mx)$$

Substituting the last equation in 10.3, we get for the velocity of progress:

$$V = \sqrt{\frac{\nu}{\nu}} \sqrt{\frac{m^2}{m^2 - \ell^2}} = \sqrt{\frac{gh}{1 - s^2}}$$  \hspace{1cm} (14.3)

where:

$$s = \frac{\ell}{m}$$

Introducing this value of $V$ in 13.3, we get:

$$u = \sqrt{\frac{g^2}{h}} \sqrt{\frac{1}{1 - s^2}} a \sin(nt - mx)$$  \hspace{1cm} (15.3)

$$\nu = \sqrt{\frac{g^2}{h}} \sqrt{\frac{s^2}{1 - s^2}} a \cos(nt - mx)$$
According to 14.3 the velocity of progress depends upon the frequency of the wave \( n \), increasing when the frequency decreases or the period length increases. When \( n < f \), the velocity becomes imaginary, and therefore the waves of the kind here considered become unstable when the frequency of the wave is smaller than the double angular velocity with which the disc rotates.

The wave motion characterised by equations 6.3 and 9.3 define a rotary motion. To the motion in the direction of progress which may be called the longitudinal, is now added a motion along the wave-front which may be called transversal, and there is a phase difference of \( \frac{\pi}{4} \) period-length between the longitudinal and the transversal motion. If velocities of the fluid particles are represented by a central vector diagram the end points of the vectors lie on an ellipse, i.e. the hodograph of the motion will be an ellipse. The ratio between the minor axis and the major axis, i.e. the ratio between the minimum and maximum velocities is:

\[
\frac{s}{c} = \frac{c}{n}
\]

and the major axis, i.e. the maximum velocity, coincides with the direction of progress of the wave motion, and is reached when the wave reaches its maximum height. The scalar value of the maximum velocity is now great compared with the amplitude of the wave and the depth, viz.:

\[
\mathbf{u}_{m} = \sqrt{\frac{g}{4}} \sqrt{\frac{1}{1 - s^2}} \cdot \mathbf{a} \quad \ldots \ldots \ldots (16.3)
\]
The direction in which the velocities rotate is negative or clockwise if the direction of rotation of the disc is positive.

26. The Energy of Waves in a Rotating Fluid

We proceed to compute the energy of the inertia waves in the same method which was followed in the previous cases; but since we used the components of velocity \( \mathbf{v} \) in the derivation of our equations, instead of the velocity-potential as we used to, we will use the undeveloped formulas for the derivation of the energy of the waves also, in the following manner:

The potential energy of a column of fluid with transversal section equal to the square unit is given by:

\[
E' = g/\rho \int_0^\gamma d\gamma
\]

and the mean potential energy over a wave-length is:

\[
E_{\text{mean}} = \frac{1}{\lambda} g/\rho \int_0^\gamma \int_0^\lambda d\gamma d\lambda = \frac{1}{4} g/\rho \lambda^2
\]

The total potential energy in one wave-length is, therefore:

\[
E = \frac{1}{4} g/\rho \lambda^2 \quad \ldots \quad (17.3)
\]

In the same manner, the kinetic energy of the column of fluid is given by:

\[
T' = \frac{1}{2} \rho/\gamma \int_0^\gamma (u' + v') d\gamma
\]

and the mean kinetic energy over one wave-length is:

\[
T_{\text{mean}} = \frac{1}{\lambda} \frac{1}{\gamma} \int_0^\lambda \int_0^\lambda (u' + v') d\gamma d\lambda
\]

\[= \frac{1}{4} g/\rho \frac{1 + \lambda}{1 - \lambda^2} \lambda^2\]
therefore the total kinetic energy in one wave-length is:

\[ T' = \frac{1}{4} g \rho a \lambda \left( \frac{1 + \frac{s^2}{l^2}}{1 - \frac{s}{l}} \right) \]  

(18.3)

And, hence, the total energy, potential plus kinetic, is

\[ T + \varepsilon = \frac{1}{4} g \rho a \lambda \left( \frac{1 + \frac{s^2}{l^2}}{1 - \frac{s}{l}} \right) \]

\[ = \frac{1}{8} g \rho a \lambda \frac{1}{l - s} \]  

(19.3)

Comparing equations 17.3 and 18.3 it is readily seen that the potential energy and the kinetic energy are no more equal, as they used to be in the case of progressive waves on a non-rotating earth; but the kinetic energy is much greater than the potential energy. The physical reason for this is is clear enough. For in this case we have the effect of the rotation of the disc. This is represented in the equations of motion by the terms \( l \nu \) and \( l \omega \) which are called the components of the Coriolis force, or the components of the inertia force, in general.

The forces of inertia tend to preserve the kinetic energy. On an unlimited disc this tendency is not prevented by any rigid boundary for which reason the major part of the energy in a progressive wave on a rotating disc remains kinetic. Since the force of inertia is directed perpendicularly to the velocity, the transversal velocities are developed, and the particles of the fluid describe ellipses. The smaller the difference between the frequency of the wave and that of the inertia-oscillation, the greater is the part of the energy
which remains kinetic, and the more the orbits of the fluid particles approach circles. This is a kind of resonance phenomenon, as is readily seen.

27- The Group - Velocity of the Inertia Waves

from equation 14.3, we have for the phase-velocity of the progressive inertia waves:

\[ V = \sqrt{gh} \sqrt{\frac{m^2}{m^2 - c^2}} \]  \hspace{1cm} (14.3)

But, \[ n = mV \]

therefore:

14.3 can be written

\[ V = \frac{1}{m} \sqrt{m^2 gh + c^2} , \text{ or} \]

\[ mV = \sqrt{m^2 gh + c^2} \]  \hspace{1cm} (14.3*)

Equation 4.1 gives for the group-velocity:

\[ W = V + m \frac{dV}{dm} , \text{ or} \]

\[ W = \frac{d(mV)}{dm} \]

and from 14.3*, we get for the group-velocity of inertia waves

\[ W = \frac{gh}{V} \]  \hspace{1cm} (15.3')

We recall that the velocity of propagation of waves on the surface of shallow fluid on a non-rotating earth is:

\[ \frac{V}{\alpha} = \sqrt{gh} \]

Let this be called \( c_0 \), so that

\[ c_0 = gh \]

Substituting this in 15.3', we get:

\[ WV = c_0^2 \]  \hspace{1cm} (15.3)
This result is of great theoretical interest. It reminds one of the De Broglie's equation in wave-mechanics, which is
\[ W \nu = c^2 \]  \hspace{1cm} (16.3)
where \( W \) is the velocity of a material particle, \( \nu \) is the velocity of propagation of the wave-motion which is associated with that material particle, and \( c \) is the velocity of light.

De Broglie's equation has had a revolutionary influence on the development of modern physics; and therefore equation 15.3 may be discussed in this light.

Comparing equation 15.3 with equation 16.3, we notice that the phase-velocity \( V \) corresponds to the quantity \( \nu \) which is the velocity of propagation of the waves associated with a material particle; and the group-velocity \( W \) corresponds to the velocity of the material particle \( W \).

The velocity of light \( c \) is the natural velocity of propagation of the electromagnetic waves in their medium - the hypothetical Ether. The theory of relativity shows that this velocity is the maximum velocity that can be attained, and this led to a new geometry of the universe, in which motion takes place in such a way as to describe the longest possible "interval".

The quantity \( c \) is the natural velocity of propagation of waves relative to their medium; which is in this case a shallow fluid on a resting basin. It is indeed the maximum possible velocity with which waves can propagate in the given medium. The time interval is therefore the shortest
possible.

It, therefore, seems to the author that there is a kind of generality in this aspect of the laws of nature. And is it possible to think of the group - velocity as having a certain amount of energy being propagated with it, while the phase - velocity has a certain amount of momentum which may be taken to correspond to that important property of all moving material particles? These questions remain unsolved for the time being, and the author thinks that they are worth of attention.

28: The Rate of Transmission of Energy in Inertia Waves

Take a plane perpendicular to the undisturbed surface of the fluid considered in Art. 25, and assume it to rotate with the fluid considered, with the same velocity. The plane is oriented in such a way as to coincide all the time with the z-y plane, i.e. that it remains always perpendicular to the x-axis. We want to compute the rate at which the pressure on one side of this plane is doing work on the fluid on the other side. This will also be equal to the rate at which energy is being propagated by the wave motion in the direction of progress, as already explained in the previous similar cases.

From equation 4.3, the variable part of the pressure is:

\[ \delta p = \frac{\eta}{\rho} \left( \eta - 3 \right) \quad \text{but} \]

\[ \eta = a \sin (nt - mx) \quad \text{ ...(10.3)} \]

therefore:

\[ \delta p = \frac{a \sin (nt - mx)}{\rho} \left[ \sin (nt - mx) - 3 \right] \quad \text{ ...(10.4)} \]
We also have from (13.3):

\[ \nu = -\frac{\partial \psi}{\partial x} = R \sin(\pi t - m \lambda) \]  \hspace{1cm} (13.3)

where:

\[ R = \frac{2}{\sqrt{1 - \xi^2}} \]

But the rate at which pressure is being done on unit width of the plane is:

\[ W_{mk} = -\int_{-\frac{\pi}{2}}^{0} R \frac{\partial \psi}{\partial x} dx , \quad \text{or} \]

\[ W_{mk} = \int_{\frac{\pi}{2}}^{0} \left[ a \sin(\pi t - m \lambda) \right] \sin(\pi t - m \lambda) dx \]

\[ = \int_{-\frac{\pi}{2}}^{0} \frac{2}{\sqrt{1 - \xi^2}} a \sin(\pi t - m \lambda) \sin(\pi t - m \lambda) \]

\[ = \frac{\zeta}{2} a^2 R \sin^2(\pi t - m \lambda) - \frac{\zeta}{2} a^2 R \sin(\pi t - m \lambda) \]

and the mean work over a complete period-length is:

\[ W_{\text{mean}} = \frac{1}{T} \int_{0}^{T} W_{mk} dt \]

\[ = \frac{1}{2} \frac{\zeta}{6} a^2 R \]

Therefore the total work done during a complete period is

\[ W_r = \frac{1}{2} \frac{\zeta}{6} a^2 R \lambda \]

\[ = \frac{1}{2} \frac{\zeta}{6} a^2 g \frac{\lambda}{\sqrt{1 - \xi^2}} \]

From formula (19.3), we have:

\[ \overline{E_r} = T + \overline{E} = \frac{1}{2} g R \frac{1}{\sqrt{1 - \xi^2}} \]

where \( \overline{E_r} \) is the total energy per one period-length; therefore:

\[ W_r = \frac{\zeta}{V} \overline{E_r} \]
We also have from formula 15.3, for the group - velocity:

\[ \omega = \frac{C}{\nu} \]

Therefore

\[ \omega_r = \omega C_r \]

which shows that the energy is propagated at a rate equal to the group - velocity, which is the same result we have been getting in all the previous cases.
CHAPTER IV

LARGE - SCALE OSCILLATIONS OF THE ATMOSPHERE

29- In this chapter, such large-scale oscillations of the atmosphere as the perturbations in the westerlies, will be treated. This problem was first treated by Rossby (Ref.18) who discussed the effect of the latitudinal variation of the Coriolis force on the propagation of oceanic and atmospheric disturbances. His results were later extended by Haurwitz who considered the effect of friction, external solinoidal force and the sphericity of the earth, (Ref.8).

We will first summarise the results obtained by Rossby and his collaborators, and then extend them according to the theory of the group-velocity. For it seems to the author that a trough in the westerlies can never be considered as an individual train of progressive sinuous waves; but it is much nearer to actuality to consider it as the resultant of a group of such waves. The following is summarised from Haurwitz's paper.

30- If the vertical variations of $u$ and $v$ are neglected, and the motion is assumed to be free from vertical accelerations, the equations of motion can be written in the following form:

\[ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - f v = - \frac{1}{\rho} \frac{\partial p}{\partial x}, \quad \ldots \quad (1.4) \]

\[ \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + f u = - \frac{1}{\rho} \frac{\partial p}{\partial y}, \quad \ldots \quad (2.4) \]

\[ \frac{\partial q}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial y}, \quad \ldots \quad (3.4) \]
where \( l = 2\omega \sin \varphi \), as before.

Assuming baratropic conditions, i.e. \( \sigma = \sigma(p) \)
and differentiating \( 2\varphi \) partially with respect to \( x \), and \( l \) partially with respect to \( y \), we get:

\[
\frac{d\xi}{dt} + (u \frac{\partial \xi}{\partial x} + v \frac{\partial \xi}{\partial y}) \cdot \dot{\xi} + (\xi + \dot{\xi})(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0,
\]

where:
\[
\frac{d}{dr} = \frac{\partial}{\partial x} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y},
\]
and:
\[
\xi = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}
\]

the vertical component of vorticity.

The last equation finally reduces to:

\[
\frac{d}{dt} (\xi + L) + (\xi + L)(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}) = 0 \quad \cdots \cdots (4.4)
\]

The equation of continuity for an incompressible fluid is:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0
\]

and assuming the motion to be purely horizontal, it reduces to:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0
\]

The total motion is composed of the undisturbed motion plus the perturbation motion, hence:

\[
\dot{u} = \dot{U} + u', \quad \dot{v} = \dot{v}', \quad \text{and} \quad \dot{w} = 0
\]

Where \( U \) is the velocity of the undisturbed westerly flow, the unprimed letters denote the total motion, and the primed the perturbation motion.

Substituting these values in (4.4) and neglecting terms of second order of magnitude, we get:

\[
\left( \frac{\partial}{\partial t} + \dot{U} \frac{\partial}{\partial x} \right) \xi + \dot{v}' \beta = 0 \quad \cdots \cdots (5.4)
\]
where
\[ \beta = \frac{\partial \omega}{\partial y} = \frac{2\omega \cos \varphi}{R} \]

and \( R \) is the radius of the earth.

Introducing a stream function \( \psi \), such that:
\[ u' = -\frac{\partial \psi}{\partial y} \quad ; \quad v' = \frac{\partial \psi}{\partial x} \]

equation 5.4 becomes:
\[ \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi + \beta \frac{\partial \psi}{\partial x} = 0 \]

where:
\[ \nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \]

Assume a solution of 6.4 in the form:
\[ \psi = C \cos m(x - vt) \cos \frac{2\pi}{D} y \]

where \( D \) is the width of the disturbance. And substituting in 5.4, we get:
\[ V = \nabla - \frac{\beta}{4\pi} \frac{\lambda}{\lambda^2 + D^2} \]

If the motion is assumed to be independent of the \( y \)-direction, the solution can be assumed in the form:
\[ \psi = C \cos m(x - vt) \]

And the formula for the velocity will be:
\[ V = \nabla - \frac{\beta}{4\pi} \frac{\lambda}{\lambda^2} \]

31-The Group Velocity

Rossby's equation gives the phase-velocity of a train of sinuous waves, and, as such, cannot be applied to the motion of troughs and ridges as they appear on the daily or
mean upper air maps. For we know that pure individual sinuous waves are not observed in the atmosphere. For a better approximation to what is taking place in reality we shall assume these troughs and ridges to be the resultant of a group of sinuous waves which are of the same amplitude and of nearly the same wave-lengths. This assumption, however, cannot be totally justified, except in the rather infrequent cases of uniformly recurring troughs and ridges, which are approximately similar in shape and in length; and in amplitudes. The more complicated case of non-uniform groups-of-waves will be discussed later.

Under the present assumptions our equation for the group-velocity is directly applicable here.

The group-velocity is given by equation 5.1, viz

\[ \mathcal{W} = \mathcal{V} - \lambda \frac{d\mathcal{V}}{d\lambda} \] \hspace{1cm} (5.1)

And from 9.4

\[ \mathcal{V} = \mathcal{V} - \frac{\beta^2}{4\pi^2} \] \hspace{1cm} (9.4)

Hence, assuming, as we already did, that the waves are progressing from west to east, so that the wave-length is independent of the latitude and \( \beta \) could be considered a constant with respect to \( \lambda \), we get:

\[ \mathcal{W} = \mathcal{V} + \frac{\beta^2}{2\pi^2} \] \hspace{1cm} (10.4)

Taking equation 7.4 instead of 9.4, we get for the group-velocity

\[ \mathcal{W} = \mathcal{V} + \frac{\beta^2}{4\pi^2} \frac{D^2}{\lambda^2} \frac{D^2}{\lambda^2} \frac{D^2}{\lambda^2 + D^2} \] \hspace{1cm} (11.4)
In formula 10.4, since $\beta$ is positive, $W$ is positive as long as we are assuming a prevailing westerly flow, i.e. as long as $U$ is positive. This would show that troughs in the westerlies have to travel always towards the east, and they can never travel in the opposite direction or even remain stationary. The velocity with which such disturbances travel is greater than the velocity of the undisturbed westerly current by an amount depending upon their wave-lengths, and varying with the square of the wave-length. The group-velocity is indeed greater than the phase-velocity. However this does happen in many physically possible cases, the best examples of which are the capillary water-waves mentioned in Art. 5.

The main objection to the results obtained in formula 10.4 is the fact that troughs are, not infrequently, observed to move towards the west. And the only way this could happen according to formula 10.4, is to make $U$ negative. This means that the zonal index should be negative to provide for a retrograde motion. To get a stationary trough $U$ has to be negative also, and it should have an absolute value of:

$$|U| = \frac{\beta \lambda^2}{4\pi^2 r^*} .$$

An interesting assumption, which does not seem very impossible at high latitudes, is that a group of waves travelling around the earth and preserving its shape and wave-length may return to its original position and interfere with a young group of waves initiated on the same system. It is clear that the resultant group will depend upon the relative
phase of the two groups, and it is unlikely that these two component groups will agree in phase in the same sense as the sinusoidal waves agreed in phase and caused their formation. However this assumption may explain why we may expect westward motion on the basis of Rossby's formula. If the two groups assumed here happened to coincide in phase then it is easily seen, by applying equation 5.1 to formula 10.4, that the resulting group will have the same velocity as the original pure individual sinuous waves, i.e., the group-velocity will be equal to the phase-velocity.

Now let us consider equation 11.4. We readily see that this equation is in a much better condition to explain the observed facts about the motion of troughs in the westerlies. For, according to this equation, a trough will move eastwards as long as $W$ is positive, i.e., as long as:

$$U + \frac{\beta x^2}{4 D^2} \frac{D^2}{x^2 + D^2} - \frac{\theta x^2}{2 D} \frac{D^2}{(x^2 + D^2)^2} = 0 \ldots \ldots \ldots (12.4')$$

This does not imply that the zonal index should be negative, in fact if we set

$$U = 0$$

then $W$ will only be zero when:

$$\lambda = D$$

otherwise, $W$ will still be positive as long as:

$$D > \lambda \ldots \ldots \ldots \ldots (12.4)$$

Formula 12.4 gives the important result:

"that, everything else being equal, a trough moves faster towards the east the more it extends in the meridional direction
i.e. the more its breadth is." If the breadth is infinite, then: $D = \infty$, and $\lambda$ is assumed to be finite, equation 12.4 shows that the trough cannot move towards the west. This result agrees with the result obtained from the simplified formula 10.4.

Since there the wave is assumed to extend from pole to pole, that is to have an infinite width.

Going back to equation 11.4, we may compute the wavelength of a stationary trough of width $D$.

For if

$$W = 0$$

then

$$\mathcal{U} + \frac{3\lambda^2}{4n^2} \lambda^2 + \frac{3\lambda^4}{2n^2 (\lambda^2 + D^2)} = 0$$

from which we get:

$$(4\pi^2 \mathcal{U} - \beta D^2) \lambda_1^2 + (8\pi^2 \mathcal{U}D + \beta D^4) \lambda_2^2 + 4\pi^2 \mathcal{U}D = 0 \quad \text{or}$$

$$\lambda_1^2 = -\frac{8\pi^2 \mathcal{U}D + \beta D^4}{8\pi^2 \mathcal{U} - 2.3D^2} \pm \sqrt{\left(\frac{8\pi^2 \mathcal{U}D + \beta D^4}{8\pi^2 \mathcal{U} - 2.3D^2}\right)^2 \frac{4\pi^2 \mathcal{U}D}{4\pi^2 \mathcal{U} - 3D^2}} \quad \text{or}$$

$$\lambda_2^2 = -\frac{8\pi^2 \mathcal{U}D + \beta D^4}{8\pi^2 \mathcal{U} - 2.3D^2} + \frac{D^3 \sqrt{3P^2 + 32\pi^2 \mathcal{U}D}}{8\pi^2 \mathcal{U} - \beta D^2}$$

and

$$\lambda_3 = \sqrt{\frac{D^3 \sqrt{3P^2 + 32\pi^2 \mathcal{U}D}}{8\pi^2 \mathcal{U} - \beta D^2} - \frac{8\pi^2 \mathcal{U}D + \beta D^4}{8\pi^2 \mathcal{U} - 3D^2}} \quad \ldots \quad \text{(13.4)}$$

Taking the positive sign under the radical, we see that in a stationary system, when $\lambda_3$ is real:

$$\frac{D^3 \sqrt{3P^2 + 32\pi^2 \mathcal{U}D}}{8\pi^2 \mathcal{U} - \beta D^2} \geq \frac{8\pi^2 \mathcal{U}D + \beta D^4}{8\pi^2 \mathcal{U} - 3D^2}$$

or:
\[(8\pi^2 \nu - 2\beta \beta^2) \left[ \frac{D^3}{\sqrt{\beta D^2 + 32\pi^2 \nu \beta}} \right] \geq 0 \]

From \[8\pi^2 \nu - 2\beta \beta^2 \geq 0\]
we get \[D^2 \leq \frac{4\pi^2 \nu}{\beta}\]
or \[|D| \leq 2\pi \sqrt{\frac{\nu}{\beta}}\]

Which is absurd, since the nature of D does not put an upper limit for it.

And from \[D^3 \sqrt{\beta^2 D^2 + 32\pi^2 \nu \beta} - (8\pi^2 \nu D^2 + \beta D) \geq 0\]
we get:
\[|D| \geq 2\pi \sqrt{\frac{\nu}{\beta}}\]

Putting the two brackets positive or negative at the same time gives the same results.

In the critical case when \[\lambda = 0\]
we get:
\[|D| = 2\pi \sqrt{\frac{\nu}{\beta}} \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots (14.4)\]

Equation 14.4 may be used to forecast the minimum width of a trough from the assumed value of the zonal index \(U\).

Finally a word might be said about the two solutions of the differential equation 6.4; the solution given by Rossby and that given by Haurwitx. The first led to formula 10. and thus to results which can hardly be explained in a satisfactory way. While the second led to formula 11.4, which, in addition to the fact that it gives no difficulty in explaining
the observed behaviour of troughs and ridges, it leads to some very useful forecasting tools like those obtained from formulas 12.4 and 14.4.

33-The Energy Of Progressive Waves in The Westerlies

Consider a train of waves progressing in the westerly current. We have from Art.30 for this wave motion:

$$\mathbf{r} = a \cos (x - vt) \cos \frac{2\pi}{D} y$$

$$u = -\frac{\partial r}{\partial y} = \frac{2ma}{D} \cos (x - vt) \sin \frac{2\pi}{D} y, \quad \text{and} \quad (15.4)$$

$$v = \frac{\partial r}{\partial y} = -a \sin (x - vt) \cos \frac{2\pi}{D} y$$

And it is readily seen from the form of the stream function that the displacement of the particles of the initial motionless surface could be represented, to a sufficient degree of accuracy, by the sinuous function:

$$\eta = a \cos (x - vt) \quad (16.4)$$

To calculate the energy of the fluid under consideration, the incompressible atmosphere, we take two horizontal planes parallel to the x-y plane, at unit distance apart.

For a unit volume of the fluid we have:

The potential energy is equal to the work done in displacing this unit volume from its position of equilibrium to a distance $y$.

Now the work is being done to overcome the resistance of the force of the pressure-gradient, which per unit volume, is equal to:

$$-\frac{\partial f}{\partial y}$$
From the geostrophic wind relation, assuming no accelerations, we have:

\[-\frac{\partial P}{\partial y} = \rho \ell \nu\]

therefore the potential energy of the unit volume of the fluid is:

\[\rho \ell \nu y.\]

The Coriolis parameter \(\ell\) is not a constant, it is assumed here to be a function of \(y\) only, i.e.

\[\ell = \ell(y) .\]

The exact nature of this functional relationship could be found by transforming our equations to polar coordinates, thus, since

\[\ell = 2 \omega \sin \varphi,\]

\[y = R \varphi , \quad R = \text{radius of earth},\]

\[dy = R d\varphi,\]

therefore the potential energy of the unit volume is:

\[E' = \rho \nu \int dy \ell \nu R \varphi\]

A unit tube of the fluid extending from the position of equilibrium to the periphery of the wave, i.e., of length \(\eta\), will have a potential energy of:

\[E'' = \rho \nu \int_0^\eta dy \ell \nu R \varphi\]

We can further transform the displacement \(\eta\) to polar coordinates by the following substitution:

\[\eta = R \alpha = a \cos m(\nu t - \alpha)\]

where \(\alpha\) is the polar angle subtended by \(\eta\).
so that the potential energy of a unit tube is:

\[ E'' = \rho V \int_{\lambda}^{R} 2 \omega \sin \varphi \cdot R \sin \varphi d \varphi \quad \text{(for simplicity let } \zeta = 0) \]

\[ = 2 \rho V w R \left( \frac{\omega}{\lambda} - \lambda \cos \lambda \right) \quad \text{or} \]

\[ E'' = 2 \rho V w R \left\{ \left[ \sin \left( \frac{2}{R} \cos (x - \lambda \theta) \right) - \frac{2}{R} \cos (x - \lambda \theta) \cos \left( \frac{2}{R} \cos (x - \lambda \theta) \right) \right] \right\} \]

Integrating over one complete wave-length, we get the following expression:

\[ E = 2 \rho V w R \int_{0}^{\lambda} \left\{ \sin \left[ \frac{2}{R} \cos (x - \lambda \theta) \right] - \frac{2}{R} \cos (x - \lambda \theta) \cos \left( \frac{2}{R} \cos (x - \lambda \theta) \right) \right\} dx \]

Now:

\[ \int_{0}^{\lambda} \sin \left[ \frac{2}{R} \cos (x - \lambda \theta) \right] dx = S_{1} = \lambda \sum_{n=0}^{\infty} \frac{(-1)^{n}}{(2n+1)^{2}} \]

Neglecting terms after the third in the series, we get:

\[ S_{1} = \frac{2 \omega}{32} \frac{\omega^{2}}{R^{2}} \lambda \]

The second part of our original integral is:

\[ S_{2} = \int_{0}^{\lambda} \left\{ \frac{2}{R} \cos (x - \lambda \theta) \cos \left( \frac{2}{R} \cos (x - \lambda \theta) \right) \right\} dx \]

\[ = 0 \]

\[ \therefore \quad E = (S_{1} + S_{2}) 2 \rho V w R \quad \text{or} \]

\[ E = \frac{2 \omega}{32} \rho V w a^{2} R \frac{\omega}{R} \lambda \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (17.4) \]

which is the potential energy of the fluid due to the wave motion.

The kinetic energy of the fluid due to the wave motion is given by:
\[ T = \frac{1}{2} \rho \int \int \int [u^2 + v^2] \, dy \, dx \]

Substituting the values of \( u \) and \( v \), we get for one period:

\[ T = \frac{1}{2} \rho \int \int \int \left[ \frac{12}{D^2} \alpha^m (x-vt) \cos \frac{2\pi}{D} y + \alpha^m \sin \frac{2\pi}{D} (x-vt) \sin \frac{2\pi}{D} y \right] \, dy \, dx \]

\[ T = \frac{\pi}{4} \rho \lambda \alpha^m \frac{2R}{\lambda} \left[ \frac{1}{D} + \frac{D}{\lambda^2} \right] \] \hspace{1cm} (18.4)

The total energy of the fluid over one period is, therefore:

\[ T + \mathcal{E} = \frac{2}{16} \rho \lambda \alpha^m \left[ \frac{1}{D} + \frac{D}{\lambda^2} \right] + \frac{21}{16} \rho \lambda U R \alpha^m \lambda \omega \] \hspace{1cm} (19.4)

34.- The Rate of Transmission of Energy in The Waves of The Westerlies

To compute the rate of transmission of energy in the waves which we are considering, we take a vertical section of the fluid, at right angles to the direction of propagation of the waves, i.e. the plane of this section is assumed to be parallel to the y-z plane. We further assume that this plane is moving with the pre-existing undisturbed current, that is, that it is moving from west to east with a constant velocity \( U \). We will now compute the rate at which the pressure on one side of this section is doing work on the fluid on the other side.

The variable part of the pressure is:
\[
\delta p = -(\eta - y) \frac{\partial p}{\partial y} = \rho(\eta - y) \ell \mathcal{U} \quad \cdot \quad \cdot \quad (20.4)
\]

therefore the work done on a strip of the section of unit height and extending to the pole, considered infinity in our present problem, is:

\[
W'_{\kappa} = - \int_0^\infty \delta p u \, dy , \quad \text{or}
\]

\[
W'_{\kappa} = - \rho \mathcal{U} \int_0^\infty \left[ a \cos(m(x - \mathcal{U} \tau) - y) \right] \ell \left[ \frac{2 \pi a}{D} \cos(m(x - \mathcal{U} \tau) \sin \frac{\pi y}{D}) \right] dy ,
\]

or, in polar coordinates:

\[
W'_{\kappa} = - \rho \mathcal{U} \int_0^{\pi} \left[ a \cos(m(x - \mathcal{U} \tau) - \mathcal{R} \phi) \right] 2 \omega \sin \phi \left[ \frac{2 \pi a}{D} \cos(m(x - \mathcal{U} \tau) \sin \frac{2 \pi \mathcal{R} \phi}{D}) \right] \mathcal{R} d\phi
\]

\[
= - \rho \mathcal{U} \mathcal{R} \int_0^{\pi/2} \frac{2 \pi a}{D} \cos(m(x - \mathcal{U} \tau) \sin \frac{2 \pi \mathcal{R} \phi}{D}) \sin \phi \mathcal{R} d\phi
\]

\[
+ \mathcal{R} \rho \mathcal{U} \int_0^{\pi} 2 \omega \sin \phi \frac{2 \pi a}{D} \cos(m(x - \mathcal{U} \tau) \sin \frac{2 \pi \mathcal{R} \phi}{D}) \phi \sin \phi \cos \phi \mathcal{R} d\phi,
\]

\[
= \left( M_1 + M_2 \right) \mathcal{R} \rho \mathcal{U}
\]

where:

\[
M_1 = - \int_0^{\pi/2} \frac{2 \pi a}{D} \omega \cos(m(x - \mathcal{U} \tau) \sin \frac{2 \pi \mathcal{R} \phi}{D}) \sin \phi \mathcal{R} d\phi d\phi
\]

Integrating over a complete period, we get:

\[
\int_0^\lambda M_1 \, dx = - \int_0^\lambda \int_0^{\pi/2} \frac{2 \pi a}{D} \omega \cos(m(x - \mathcal{U} \tau) \sin \frac{2 \pi \mathcal{R} \phi}{D}) \sin \phi \mathcal{R} d\phi d\phi d\mathcal{U}
\]

\[
= \left( \frac{2 \pi a \mathcal{U}}{4 \pi \mathcal{R}^2 - \mathcal{D}^2} \right) \omega \cos \frac{\pi \mathcal{R}}{D}
\]
Also
\[ \int_0^l \int_0^\frac{\pi}{2} \int_0^\frac{\pi}{2} \frac{\tilde{D}}{D} \cos(x - \nu t) \cdot \sin \phi \cdot \sin \frac{\pi R}{D} \, d\psi = 0. \]

The total work per unit period is, therefore:

\[ W_T = \frac{2\pi a^2 D \rho U R}{4\pi^2 R^2 - D^2} \omega \cos \frac{\pi R}{D}. \ldots \ldots (21.5) \]

The rate at which the energy is being transmitted may now be computed from the relation:

\[ W_T = (E + T) V_e, \]

where \( V_e \) is the velocity of energy.

Substituting from 19.4 and 21.4, we get:

\[ \frac{2\pi a^2 D \rho U R}{4\pi^2 R^2 - D^2} \omega \cos \frac{\pi R}{D} = \left\{ \frac{R \pi^2}{16} \rho a^2 \lambda \left[ \frac{1}{D} + \frac{D}{\lambda} \right] + \frac{21}{16} \rho U R \lambda^2 \omega \right\} V_e \]

\[ \therefore V_e = \frac{\frac{2\pi D U}{4\pi^2 R^2 - D^2} \omega \cos \frac{\pi R}{D}}{\frac{R \pi^2}{16} \left( \frac{\lambda}{D} + \frac{D}{\lambda} \right) + \frac{21}{16} U \omega}. \ldots (22.4) \]

This formula gives the "velocity of energy" relative to the undisturbed current. The velocity relative to the earth's surface will, therefore, be:

\[ V_e = U + \frac{\frac{2\pi D U}{4\pi^2 R^2 - D^2} \omega \cos \frac{\pi R}{D}}{\frac{R \pi^2}{16} \left( \frac{\lambda}{D} + \frac{D}{\lambda} \right) + \frac{21}{16} U \omega}. \ldots (22.4) \]

Comparing this formula with formula 11.4, we readily see that they are not in agreement. Therefore it appears
that the energy of a trough does not have to be propagated with the trough itself. This may be interpreted as follows: Since the velocity of a trough does not agree with the velocity of its energy, a new trough is expected to form where the energy is transported, while the old trough will die out rapidly as it loses its energy. This is often observed in the atmosphere; but we do not like to emphasise the significance of our equations, for they might be more misleading than useful. Formula 22.4 was derived after taking the spherical shape of the earth into consideration, while formula 11.4 was derived on the basis of a plane earth, with the additional slight correction for the variation of the Coriolis parameter on a spherical earth. In our derivation for formula 22.4 the definite integrals appearing in the expression for the potential energy and those appearing in the expression for the work will hold to be definite if the earth is assumed a plane. To overcome this difficulty the previous scheme was followed.

If it comes to preferring one formula to the other then it seems to the author that formula 22.4 is more accurate, and the velocity of a trough should be computed from \( \mathfrak{D} \).

Another way of doing the evaluation of these integrals is to assume that the Coriolis parameter to be a linear function of \( y \); thus we get:

\[
\ell = \ell_0 + \beta y
\]

This hardly justifiable. For if we take the more general expression for the functional dependence of the Coriolis
parameter on the ordinate $y$, and expand it by a Taylor series we get:

$$
\ell(y) = \ell_0 + \left( \frac{2 \ell}{\eta \eta_0} \right) y + \frac{1}{2!} \left( \frac{2 \ell}{\eta \eta_0} \right)^2 y^2 + \frac{1}{3!} \left( \frac{2 \ell}{\eta \eta_0} \right)^3 y^3 + \ldots
$$

To assume a linear relationship is in effect to neglect terms of second and higher degrees in this series; and this is permissible only when $y$ is small. For the second derivative of $\ell$ is equal to $\ell$ in magnitude, and therefore it cannot be neglected. Therefore a linear relationship can be assumed when the wave does not extend appreciably in the meridional direction.

Considering such an approximation and proceeding in the derivation as before, we get; after neglecting terms of the second and higher powers:

$$
\ell = \ell_0 + \beta \eta
$$

And assuming $\beta$ to be constant, we get for the potential energy of a unit tube:

$$
E'' = \rho U \int_0^\eta \left( \ell_0 + \beta \eta \right) \eta \, d\eta
$$

$$
= \frac{1}{2} \rho U \eta^2 + \frac{1}{3} \rho U \beta \eta^3
$$

$$
= \frac{1}{2} \rho U \alpha^2 \cos^2 m(x-Vt) + \frac{1}{3} \rho U \beta \alpha^3 \cos^3 m(x-Vt).
$$

Integrating over a complete period we get:

$$
E = \frac{1}{2} \rho U \alpha^2 \int_0^\lambda \cos^2 m(x-Vt) \, dx + \frac{1}{3} \rho U \beta \alpha^3 \int_0^\lambda \cos^3 m(x-Vt) \, dx.
$$

But since $\cos^3 \theta$ is an odd function, its average
over a complete period is zero; so, we get for the potential energy of a complete wave-length:

\[ E = \frac{1}{4} \rho \lambda \varepsilon \alpha^2 \lambda \]  

(17.4')

For the kinetic energy, we get the same expression as before, that is:

\[ T = \frac{RN^2}{16} \rho \lambda \alpha^2 \left( \frac{1}{\beta} + \frac{D}{\lambda} \right) \]  

(18.4')

Therefore the total energy is

\[ T + E = \frac{RN^2}{16} \rho \lambda \alpha^2 \left( \frac{1}{\beta} + \frac{D}{\lambda} \right) + \frac{1}{4} \rho \lambda \varepsilon \alpha^2 \lambda \]

And for the work we get:

\[ W_{m'} = -\rho \varepsilon \int_0^\infty \left[ a \cos (m(x-Vt)) - \frac{2\pi m}{D} \right] u \, dy \]

\[ = -\rho \varepsilon \int_0^\infty \left[ a \cos (m(x-Vt)) - \frac{2\pi m}{D} \right] \cos (m(x-Vt) \sin \frac{2\pi y}{D}) (\beta_y) dy \]

\[ = -\rho \varepsilon \int_0^\infty \left[ \frac{2\pi m}{D} \cos (m(x-Vt)) \sin \frac{2\pi y}{D} - \frac{2\pi m}{D} \cos (m(x-Vt)) \sin \frac{2\pi y}{D} \right] \sqrt{\left( \beta_y \right)} dy \]

\[ \sqrt{(\beta + \beta_y)} dy \]

\[ = -\rho \varepsilon \int_0^\infty \left[ \frac{2\pi m^2}{D} \cos (m(x-Vt)) \sin \frac{2\pi y}{D} - \frac{2\pi m}{D} \cos (m(x-Vt)) \sin \frac{2\pi y}{D} \right] dy \]

\[ - \rho \varepsilon \int_0^\infty \left[ \frac{2\pi m^2}{D} \cos (m(x-Vt)) \sin \frac{2\pi y}{D} - \frac{2\pi m}{D} \cos (m(x-Vt)) \sin \frac{2\pi y}{D} \right] dy dy \]

or \[ W_{m'} = -\rho \varepsilon \left( S + S_2 \right) \]
where: \[ S_1 = \lambda \int_0^\infty \left[ \frac{2\pi a^2}{D} \cos (x - \nu t) \sin \frac{2\pi y}{D} - \frac{2\pi a^2}{D} \cos (x - Vt) \sin \frac{2\pi y}{D} \right] dy \]
\[ = \lambda a^2 \cos (x - \nu t) + M_1 \]

Integrating over a complete period, we get:
\[ \int_0^\lambda S_1 dx = \frac{1}{2} \lambda a^2 \lambda, \]

since:
\[ \int_0^\lambda M_1 dx = \int_0^\lambda \left[ -\frac{2\pi a^2}{D} \cos (x - \nu t) \int_0^\infty \sin \frac{2\pi y}{D} dy \right] dx = 0 \]

Also:
\[ S_2 = \beta \int_0^\lambda \left[ \frac{2\pi a^2}{D} \cos (x - \nu t) \sin \frac{2\pi y}{D} dy - \frac{2\pi a^2}{D} \cos (x - Vt) \sin \frac{2\pi y}{D} dy \right] \]
\[ = \frac{1}{2} \beta a^2 \cos m (x - \nu t) \left[ \sin \frac{\pi R}{D} - \frac{\pi R}{D} \cos \frac{\pi R}{D} \right] + M_2 \]

where:
\[ M_2 = -\beta \frac{2\pi a^2}{D} \cos (x - \nu t) \int_0^\infty y \sin \frac{2\pi y}{D} dy \]

Integrating over a complete period, we get:
\[ \int_0^\lambda S_2 dx = \frac{a^2 D \beta}{4 \pi^2} \lambda \left[ \sin \frac{\pi R}{D} - \frac{\pi R}{D} \cos \frac{\pi R}{D} \right], \]

since, again:
\[ \int_0^\lambda M_2 dx = 0 \]

The total work over a complete period is, therefore
\[ W = \left[ \frac{1}{2} \lambda a^2 \lambda + \beta a^2 \frac{D}{4 \pi^2} \lambda K \right] \rho U, \text{ or} \]
\[ W = \frac{1}{2} \rho U \lambda a^2 \lambda + \rho U \beta a^2 \frac{D}{4 \pi^2} \lambda K \quad (20v) \]

where:
\[ K = \sin \frac{\pi R}{D} - \frac{\pi R}{D} \cos \frac{\pi R}{D} \]
The rate at which the energy is being transmitted could be found from the relation:

$$W' = (E + T) V_e'$$

But, from 19.4, we have:

$$E + T = \frac{R \pi^2}{16} \rho \lambda a^2 \left( \frac{1}{D} + \frac{D}{\lambda^2} \right) + \frac{1}{4} \rho \nu \xi_a a^2 \lambda$$

so that:

$$V_e' = V \frac{\frac{1}{4} \xi_0 + \nu \beta \frac{D}{4 \pi} \kappa}{\frac{R \pi^2}{16} \left( \frac{1}{D} + \frac{D}{\lambda^2} \right) + \frac{1}{4} \nu \xi_a} \ldots (22.4')$$

It is not wise to emphasise the importance of this formula, for in deriving it, some approximations were made which are hardly justifiable.

It is not possible right of hand to put formula 22.4' in the same shape as that of either 10.4 or 11.4; for this can only be done after knowing the value of $K$. However, it is seen that this formula does not agree with either of those two. This simply shows that the conclusions derived from the more exact solution given in the beginning of this article are to be taken more seriously. For it is now found that the trough and the energy associated with it may have different velocities. And this is the same thing we found from the first solution.

It might be mentioned again that these results, in spite of their agreement with the observed facts, should not be tried quantitatively for the basic reason of the approximations.
which were introduced in the derivation of formula 22.4', and which were mentioned at the beginning of the solution.

A more accurate representation of the whole problem will be attempted later, after introducing a new concept of the group - velocity which may suit the cases of irregular waves better than the present concept.
CHAPTER V

GENERALIZED CONCEPT OF THE GROUP - VELOCITY

35- General Considerations

The concept of the group - velocity as given in the previous chapters, and as developed by Lord Rayleigh, applies directly to waves motions in which the component waves are sinuous in shape, and all the different trains have nearly equal amplitudes and nearly equal wave-lengths. Such waves can only produce groups of waves which are periodic in themselves, and which finally can be considered as regular deformations in the medium in which they are being propagated. In treating atmospheric disturbances one rarely observes waves that repeat themselves in such an accurate periodicity, which the previous methods require, in order to give results with a good degree of accuracy. The cases treated in the previous chapters were all cases of sinuous regular waves, which are more or less of hypothetical nature. It is, therefore, felt that a new concept of the group-velocity, if not a new method of analysis, is needed to justify the treatment of such phenomena as troughs produced by a concentrated solfnoidal field, or frontal waves and occluded frontal systems, where the disturbed surfaces are far from being described as regular waves.

The following discussion, which seems to be best fitted for the purpose, is mainly taken from a theory which was developed by Lord Kelvin (Ref.11), and later by Green$ (Ref6).
36- Kelvin-Green's Theory

In his paper on "The Waves Produced by a Simple Impulse in a Dispersive medium", Lord Kelvin treated the case of a group of waves of irregular shape propagated in a dispersive medium, where the velocity of the waves is a function of the wave-length. He says in the beginning of his paper:

"The results of our work will show us that the velocity of progress of a zero, or a minimum, or a maximum, in any part of a varying group of waves is equal to the velocity of progress of periodic waves of wave-lengths equal to a certain length, in the neighbourhood of the particular point looked to in the group (a length which will generally be intermediate between the distance from the point considered to its next-neighbour corresponding points on the preceding and following waves)"

This idea was later carried on by Green and on its basis he arrived at a more general concept of the group-velocity.

The Kelvin-Green's theory may be summarised as follows:

Let \( V = f(m) \) denote the wave-velocity of an infinite train of waves of period \( T = \frac{2\pi}{\nu} \), and of wave-length \( \lambda \); the medium is supposed to be dispersive, so that each of the period and the velocity corresponding to \( \lambda \) are functions of \( m \). The Fourier synthesis gives for the displacement \( \mathbf{f} \) at any point and at any time \( t \), caused by an infinitely intense disturbance at the origin at zero time:
Thus according to this equation, the initial disturbance may be regarded as composed of an infinite number of regular waves of equal amplitude, all of which agree in phase at the origin. At any time after the commencement of the disturbance, the shape of the deformed surface may be determined by the trains of waves which agree in phase, the remaining trains are supposed to be of all possible phases so that they may be assumed to cancel each other by their mutual interference. The trains of waves that agree in phase may be assumed to have an infinite number. The mean period of the effective trains at each point of the medium, and the phase at which this agreement occurs vary continuously. We therefore speak of a certain wave-period which is the mean of all the trains whose coincidence in phase at any point determines the displacement of the medium at that point, as the predominant period at that point; and our problem is to determine at what period of the medium any specified wave-period will be the predominant period at any time.

Returning now to the equation 1.5, we see that the parts of the integral which lie on the two sides of a small range, \( \mu - \alpha \) to \( \mu + \alpha \), vanish by mutual interference; \( \mu \) being the value of \( m \) which makes

\[
\frac{d}{dm} \left[ m (\xi - tf(m)) \right] = 0 \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (2.5)
\]
so that:  \[ m[\chi - t f(m)] = \mu \frac{\sigma \sqrt{2}}{\epsilon^2 \frac{1}{\epsilon^2} \left[-m f''(m) - 2f'(m)\right]^{\frac{1}{2}} (m - m)\frac{1}{2}} \]  \[ \text{(3.5)} \]

2.5 may be written as:  \[ \chi - t \left[f(m) + m f'(m)\right] = 0 \]  \[ \text{(4.5)} \]

and by Taylor's series, for \( m - m \) very small, we have:  \[ m[\chi - t f(m)] = \mu \left[\chi - t f(m)\right] - t\left[m f''(m) + 2f'(m)\right] \left[m - m\right] \]  \[ \text{(5.5)} \]

Substituting from \( \beta.5 \), we get:  \[ m[\chi - t f(m)] = \frac{t}{\epsilon^2 \frac{1}{\epsilon^2} \left[-m f''(m) - 2f'(m)\right]^{\frac{1}{2}}} \left(m - m\right) \]  \[ \text{(6.5)} \]

or  \[ m - m = \frac{\sigma \sqrt{2}}{\epsilon^2 \frac{1}{\epsilon^2} \left[-m f''(m) - 2f'(m)\right]^{\frac{1}{2}}} \]  \[ \text{(7.5)} \]

where:  \[ \sigma^2 = m[\chi - t f(m)] - \mu t f'(m) \]

And from 1.5, we get:  \[ \mathcal{F} = \frac{\int_{-\infty}^{\infty} c_0 \left[\epsilon f' + \sigma^2\right] d\epsilon}{2 \epsilon^2 \frac{1}{\epsilon^2} \left[-m f''(m) - 2f'(m)\right]^{\frac{1}{2}}} \]  \[ \text{(8.5)} \]

upon integrating 8.5, it becomes:  \[ \mathcal{F} = \frac{c_0 \left[\epsilon f' + \sigma^2\right]}{2 \epsilon^2 \frac{1}{\epsilon^2} \left[-m f''(m) - 2f'(m)\right]^{\frac{1}{2}}} \]

Or, since by 3.5  \[ \epsilon f' = \mu [\chi - t f(m)] \]

\[ \therefore \mathcal{F} = \frac{c_0 \left[\mu \left[\chi - t f(m)\right] + \frac{\sigma^2}{\epsilon^2}\right]}{2 \epsilon^2 \frac{1}{\epsilon^2} \left[-m f''(m) - 2f'(m)\right]^{\frac{1}{2}}} \]  \[ \text{(9.5)} \]

Returning now to equation \( \beta.5 \), we note:
\[
\frac{\partial \xi}{\partial t} = \left[ f(m) + mf'(m) \right]
\]

and setting

\[
\frac{\partial \xi}{\partial t} = \omega,
\]

we get:

\[
\omega = f(m) + mf'(m)
\]

...(10.5)

where \( \omega \) is the group-velocity. Remembering that \( f(m) = V \)

10.5 becomes:

\[
\omega = V + m \frac{dV}{dm}
\]

...(11.5)

This result is the same as the one which was

given in Chapter I, and which was used throughout the preceding cases; but we have now a different concept of the group-velocity which is more general than the previous one. For according to the new concept it is no more necessary to have

sinuous trains of waves of nearly the same length, as long as we understand by the group-velocity: the velocity of a certain point on the wave where a certain wave-length \( \lambda \) is to be found. In a group of waves progressing in a dispersive medium and arising from an initial disturbance, the different components of the group will move each according to its own velocity, and as time goes on, the different components will be gradually sorted out and the shape of the deformed surface will approach sinusoidality more closely, and if by a group we understand, after Lamb, "a succession of waves in which the distance between successive crests varies slightly", then if a certain wave-length \( \lambda \) is observed in a group at a place \( x \) at time \( t \)
this particular wave-length will be found after a time $\Delta t$ at a
place $x'$ given by the equation:

$$x' = x + W\Delta t$$

where $W$ is the group-velocity corresponding to the wave-length
$\lambda$.

Thus we can speak of a group-velocity with reference to each small part of the original "group"; but the group
as a whole has no definite "group-velocity", unless its group-
velocity is defined as the mean of the group-velocities of its
parts. Such a group quickly loses its identity on account of
the dispersive effect of the medium.

This concept of group-velocity becomes especially
useful when we treat cases of irregular waves, or cases of waves
initiated by an external force concentrated at a small limited
region of the medium. For the disturbance can then be assumed
to consist of an infinite number of sinuous waves of all
possible wave-lengths, and thus it can be assumed that once
an external impulse is applied to the medium, the disturbing
effect will reach infinitely great distances after an infinit-
esimal time. Irregularities in shape will continuously occur
because of the faster waves overcoming the slower ones.
CHAPTER VI

WAVE MOTION UNDER THE INFLUENCE
OF EXTERNAL FORCES

In the previous cases considered so far the fluid was supposed to be moving under no external forces other than gravity or gravity and pressure-gradient; and the results obtained so far can only be applicable if this condition is realized.

Wave motion in the atmosphere is, however, rendered much more complicated by the fact that external forces, other than those mentioned, are always present; and to get any results comparable to the observed motion these forces should be taken into consideration. In the present chapter external forces like friction at the earth's surface, and the action of solenoidal fields will be treated. Forces acting for a short period at a limited extent, of the nature of momentary impulses, will also be considered. For we may think of a wave disturbance being initiated at a surface of discontinuity by the influence of an outside force acting for a short period. Such cases might be taken to represent wave motion on a stationary frontal surface when such a surface has acted upon by a rapid change of pressure, or by an isallobaric force.
37- Perturbation of The Westerlies When Friction Is Acting

In discussing wave motion in the Westerlies, the assumption was made that no external forces of any kind are acting on the fluid. In the present article the case of outside friction, i.e. friction with the ground, will be considered. This was first treated by Haurwitz in his paper "On Motion of Atmospheric Disturbances" (Ref. 8), to which reference has already been made. In the present article a summary of this paper will first be given, and then it will be extended in view of the theory of the group-velocity.

The frictional force will be assumed to be proportional to the wind velocity and to be acting in the opposite direction. Thus:

\[ \vec{F} = -k \vec{V} \]  \hspace{1cm} (1.6)

where \( \vec{F} \) is the frictional force, \( \vec{V} \) is the wind velocity, and \( k \) is a constant of proportionality known as the coefficient of friction. The minus sign indicates that the two vectors are opposite in direction.

Now, the vertical component of vorticity \( \zeta \) is given by:

\[ \zeta = \nabla \times \vec{V} \]

Multiplying both sides by \( -k \), we get:

\[ -k \zeta = -k \nabla \times \vec{V} \]

\[ \therefore -k \zeta = \nabla \times (-k \vec{V}) = \nabla \times \vec{F} \]  \hspace{1cm} (2.6)

which shows that the vertical component of vorticity is proportional to the curl of the frictional force.

The equation of relative motion, in vector form,
may be written:
\[
\frac{d\mathbf{v}}{dt} + 2 \mathbf{\hat{\omega}} \times \mathbf{v} = \mathbf{\Gamma}^\prime - \nabla \left( \frac{\rho}{\xi} \right) \tag{3.6}
\]
where \( \mathbf{\hat{\omega}} \) is the angular velocity of the earth's rotation, and the other symbols have their usual meanings.

operating on both sides of equation 3.6 by \( \nabla_x \), we get:
\[
\nabla_x \times \left( \frac{d\mathbf{v}}{dt} \right) + \nabla_x \times \left( 2 \mathbf{\hat{\omega}} \times \mathbf{v} \right) = \nabla_x \mathbf{\Gamma}^\prime - \nabla_x \left[ \nabla \left( \frac{\rho}{\xi} \right) \right]
\]
and since the differential operators are commutative, this becomes:
\[
\frac{d}{dt} \left( \nabla_x \mathbf{v} \right) + \nabla_x \times \left( 2 \mathbf{\hat{\omega}} \times \mathbf{v} \right) = \nabla_x \mathbf{\Gamma}^\prime - \nabla_x \left[ \nabla \left( \frac{\rho}{\xi} \right) \right]
\]
Transferring back to scalar representation, and making use of formulae 1.6 and 2.6, we get:
\[
\frac{d\Psi}{dt} + \kappa \Psi + \beta \nu = 0 \tag{4.6}
\]
As in the previous cases, the total motion may be considered as composed of the undisturbed motion and the perturbation motion. Substituting in 4.6 and neglecting quantities of second order of magnitude, we get:
\[
\left( \frac{\partial}{\partial t} + \mathbf{\nabla} \cdot \mathbf{v} + \kappa \right) \left( \frac{\partial \mathbf{v}'}{\partial x} - \frac{\partial \mathbf{v}'}{\partial y} \right) + \beta \nu' = 0 \tag{5.6}
\]
The equation of continuity for an incompressible fluid in two-dimensional form, and for the case of perturbation motion is, as before:
\[
\frac{\partial \mathbf{v}'}{\partial x} + \frac{\partial \mathbf{v}'}{\partial y} = 0 \tag{6.6}
\]
Introducing a stream function \( \psi' \), this becomes:
\[
\left( \frac{\partial}{\partial t} + \mathbf{U} \cdot \nabla + \kappa \right) \left( \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y} \right) + \beta \frac{\partial \psi}{\partial x} = 0 \quad \cdots \quad (7.6)
\]

A solution is assumed in the form:
\[
\psi = A \mathcal{N} e^{i(mx - ny)} e^{i\xi y} \quad \cdots \quad (8.6)
\]

where \( \xi = \frac{\omega}{D} \).

Substituting 8.6 in 7.6, we get:
\[
\mathcal{N} = m \left( \mathbf{U} - \frac{\beta \lambda}{2 \mu + \alpha^2} \right) - i\kappa \quad \cdots \quad (9.6)
\]

Substituting back in 8.6, we get for the stream function:
\[
\mathcal{N} = A e^{\omega m} \left[ \kappa - \left( \mathbf{U} - \frac{\beta \lambda}{2 \mu + \alpha^2} \right) t \right] e^{-\kappa t} e^{i\omega m} \frac{\omega}{D} y \quad \cdots \quad (10.6)
\]

Assuming that the initial form of the stream function is:
\[
\mathcal{N} = C' \quad \text{a const., } \quad \psi = 0
\]

and generalizing 10.6 by the generalized Fourier's theorem, we get:
\[
\mathcal{N} = \frac{C}{m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\kappa t} e^{i\omega m} \left[ \kappa - \left( \mathbf{U} - \frac{\beta \lambda}{2 \mu + \alpha^2} \right) t \right] dm \, ds \quad \cdots \quad (11.1)
\]

where \( C \) is equal to \( C' A = \text{a const.} \),

or more generally:
\[
\mathcal{N} = \frac{C}{m} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-\kappa t} e^{i\omega m} \left[ \kappa - \left( \mathbf{U} - \frac{\beta \lambda}{2 \mu + \alpha^2} \right) t \right] dm \, ds \quad \cdots \quad (11.6)
\]

From equation 10.6, it is readily seen that the phase-velocity of an individual sinusoidal wave is given by:
\[
\mathbf{V} = \mathbf{U} - \frac{\beta \lambda}{4 \mu + \alpha^2} \frac{D^t}{D^t + \lambda^t}
\]
which is the same as that for the case of non-frictional motion.

From equation 11.4, the group-velocity of this motion will be given by:

\[ \omega = \nu + \frac{\beta \lambda^2}{4 \eta} \frac{D^x}{D^y} - \frac{\beta \lambda^2}{4 \alpha^2} \left( \frac{D^x}{D^y} \right)^2 \]  \hspace{1cm} (11.4)

which is, again, the same as that for the case of non-frictional motion. However in the present case, the group of waves is taken to mean that in which a certain defined wave-length is to be found. The group may lose its definite identity after a relatively short time, to the point of being unrecognisable.

In fact it is evident from the equation of the stream function that the amplitude decrease with time due to the effect of friction, and the wave will have its maximum amplitude at the initial time \( t=0 \).

The effect of friction of friction on a pure sinuous train of waves was discussed completely by Haurwitz, in his paper where he gives a curve for the distribution of the vertical component of vorticity along the \( x \)-axis, and which is reproduced in Fig.2.

Assuming the stream lines are also isobars, equation 10.6 will give the general shape of the pressure pattern. Fig.3 shows this pattern on the assumption of no friction, and Fig.4 shows the same thing for frictional motion. These figures are computed on the assumption of simple sinuous individual waves.
38- Waves Caused By an Outside Impulse

We shall now consider the case of a group of waves formed by the interference of an infinite number of simple sinusoidal trains of waves of the same shape as those discussed in the previous article. For the sake of simplicity, it will be assumed that these waves are initiated by an infinite impulse concentrated at the origin, at time \( t = 0 \). It will further be assumed that they are all governed by the same law of propagation which is given in the previous article, that is, their velocities of propagation shall be in accordance with the formula:

\[
\nu = \nu^* - \frac{\beta}{\omega^*} \frac{\partial^2}{\partial x^2}
\]

All the interfering waves are assumed to be of the same initial amplitude, but their periods are different and may have any real value between 0 and \( \infty \).

It is clear that this case is a special case of the one discussed by Lord Kelvin, and summarized in Chap. V. Therefore the displacement of the medium at time \( t \) and at position \( x \) is given by the formula:

\[
\xi = \frac{A \epsilon_0 \epsilon \left[ \frac{1}{\nu} \left( \nu - \frac{\nu^*}{\omega^*} \right) \frac{\partial}{\partial t} \right] \left[ \frac{\partial}{\partial x} \right]}{2\pi \nu^* \epsilon \left[ \ln \left( \frac{\omega^*}{\nu^*} \right) \right] \nu^*}
\]

(2.6)

where the function \( f(m) \) is the velocity of propagation of the individual waves, and \( A \) is a constant.

Assuming that all the waves are of the same extent in the \( y \)-direction, which is to be taken as \( \nu = \frac{3}{2} \lambda \) for
example. And substituting in 12.6, we get:

\[ \xi = \frac{\alpha m\sqrt{13}}{6\sqrt{\pi t^{\frac{1}{3}}} h} \cos \left\{ m \left[ x - t \left( \frac{13}{4.5} \frac{\beta}{m} \right) + \frac{3}{4} \right] \right\} \cdots (13.6) \]

where:

The amplitude of the curve which represents the displacement is therefore given by:

\[ a = \frac{\alpha m\sqrt{13}}{6\sqrt{\pi t^{\frac{1}{3}}} h} \cdots \cdots (14.6) \]

from which it is seen that the amplitude decreases with time, even in this case where no friction is assumed to be acting; so that the oscillation is essentially a damped oscillation. The damping being caused by the dissipation effect of the medium, which acts to dissipate the energy of the initial disturbance and spread it over wider and wider areas, and thus continuously decreases the amount of energy at the group.

The author thinks that this result is quite important in explaining the behavior of troughs and ridges which are observed at higher levels where external friction is small or even negligible, and yet it is observed that these troughs and ridges are quickly damped and their amplitudes decrease rapidly as soon as they recede from the sources of energy. This fact has often been overlooked by earlier writers and the damping was usually attributed to the frictional forces at the ground.
Going back to equation 14.6, it is seen that, since \( \beta \) increases with decreasing latitude, and since the amplitude decreases with increasing values of \( \beta \), then the amplitude should be expected to decrease with decreasing latitude. Thus, everything else being equal, waves that are generated at higher latitudes will have greater amplitudes than those generated at lower latitudes. At the equator, if such waves are assumed to exist, the amplitude will have its minimum value; whereas at the poles the amplitude will have an infinite magnitude. This simply means that no waves are possible at the poles, and the only possible kind of disturbances that might take place there is that which has a straight line configuration. Mathematically, the poles are singular points in the curves of distribution of amplitude against latitude. Hence the isobars may either be concentric closed figures having the pole as their mathematical pole, a focus for example, or they may approach the pole from outside in an asymptotic manner.

Equation 14.6 further shows that the amplitude is inversely proportional to the wave-length of the predominant wave; and conversely, the wave-length of the predominant wave is inversely proportional to the amplitude. But since the amplitude decreases with time, as already pointed out, the wave-length of the predominant wave in the group should increase continuously with time. The group as a whole will, therefore, appear as decreasing in amplitude and increasing in length.

Figure 5 shows the variation of the amplitude
Fig. 34.
of a wave starting at the origin. It clearly shows the decrease of the amplitude with time. One feature of this curve is worthy of special attention: and that is that the amplitude decreases very rapidly in the first short period, and then it decreases more and more slowly approaching the zero value asymptotically. As a matter of fact it could well be considered parallel to the $f$-axis after a time of four or five units. At the region of approximate parallelism, the individual waves could be assumed as sinusoidal in shape, and the classical treatment of perturbations could be assumed approximately correct at this region of the curve. However, at this region of the curve, dispersion is so advanced that the amplitude has decreased to a small fraction of its initial magnitude. The waves could well be considered as unnoticeable in the atmosphere. For it follows from 14.6 that the amplitude decreases to $1/60$ of its value at unit time after one hour.

The above-cited analysis shows moreover that in order to maintain disturbances of appreciable displacement, as ordinarily observed in the atmosphere, it is not enough to have an impulse acting for a short period and then stopping. But rather, it is necessary that the external force should continue its action through a reasonably long period. This seems to be always the case in the atmosphere. For if the outside impulse is caused by a solenoidal field, for example, the solenoidal field does not disappear after the first impulse; but it continues to exist for a long time, thus producing a continuous cause of disturbance.
To complete the picture of atmospheric waves, Fig.6, which is reproduced from Kelvin's paper, is given here. This figure shows the general shape of the group as it proceeds in the medium. It clearly shows the dispersive effect of the medium on the wave-length and on the amplitude.

The effect of friction on the waves is to accentuate the dispersive effect of the medium on the amplitude. For in this case, the amplitude, as given by \(14.6\), has to be multiplied by the factor \(e^{-kt}\). Figure 7 shows the combined effect of dispersion and friction on the amplitude. In this figure, the initial time is taken as one hour after the initial impulse has ceased to act, and the scale of the amplitude is magnified to make the effect observable. The same thing is given in Table II. The values for the effect of friction are taken from Haurwitz's paper. The dispersive effect is computed on the basis of unit amplitude at one second after the initial impulse.

**Table II**

<table>
<thead>
<tr>
<th>Time in days</th>
<th>Reduction due to dispersion (^{-2})</th>
<th>Reduction due to friction</th>
<th>Total reduction (^{-2})</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/24</td>
<td>1.6.10         (^{-2})</td>
<td>0.99</td>
<td>1.58.10 (^{-2})</td>
</tr>
<tr>
<td>10/24</td>
<td>0.52.10 (^{-2})</td>
<td>0.84</td>
<td>0.44.10 (^{-2})</td>
</tr>
<tr>
<td>2</td>
<td>0.24.10 (^{-2})</td>
<td>0.42</td>
<td>0.1.10 (^{-2})</td>
</tr>
<tr>
<td>4</td>
<td>0.17.10 (^{-2})</td>
<td>0.18</td>
<td>0.03.10 (^{-2})</td>
</tr>
<tr>
<td>6</td>
<td>0.13.10 (^{-2})</td>
<td>0.075</td>
<td>0.0098.10 (^{-2})</td>
</tr>
<tr>
<td>8</td>
<td>0.12.10 (^{-2})</td>
<td>0.031</td>
<td>0.0037.10 (^{-2})</td>
</tr>
<tr>
<td>10</td>
<td>0.11.10 (^{-2})</td>
<td>0.013</td>
<td>0.0014.10 (^{-2})</td>
</tr>
</tbody>
</table>
It is seen from this table that dispersion is much more effective in the early stages of the disturbance when friction is hardly effective; while friction becomes more effective in the final stages when dispersion is hardly effective. The two factors are nearly equal in the second day after the initiation of the disturbance.

39-Definition of a Wave Packet

A group of waves which forms an independent pulse, by itself, outside of which the medium may be considered as undisturbed, will be called a "packet of waves". The term is borrowed from electromagnetics where its meaning is restricted by the assumption that the pulse is composed of a limited small region of the spectrum. In the latter branch of science the mathematical definition of a packet is:

\[ \psi = \int_{m_0 - i\infty}^{m_0 + i\infty} A(m) e^{i(mx - nt)} \, dm \quad (15.6) \]

where \( \psi \) is the amplitude of the resulting packet, and \( m \) and \( n \) have their usual meanings. \( A(m) \) being the amplitude of the individual waves expressed as a function of \( m \).

According to the usage of this term here, the packet may be assumed to consist of all possible wave-lengths in the spectrum, between zero and infinity, and the mathematical expression for a packet will therefore be:

\[ \mathcal{F} = \int_{0}^{\infty} A(m) e^{i(mx - nt)} \, dm \quad (16.6) \]

The same concept has been used in the previous
articles, and equation 1.5 is equivalent to 16.6.

To justify the extension of the limits, it is only necessary to notice that there is a great difference between the dispersion effect of the Ether on electromagnetic waves, and the effect of the air on atmospheric waves. For the first is a slowly dispersive medium, while the second is a much faster one. The discussion given in the last two articles, of the atmospheric waves should make this point clear and no further comments are therefore necessary.

40- On The Periodicity of The Waves in The Westerlies

It has often been expressed that a good deal of doubt exists about the wave theory of cyclones because of the lack of periodicity in these phenomena. It is the object of this article to explain this question in the light of (1) the theory of wave-groups, and (2) the analysis which has been given for the case of waves in the westerlies. This reasoning is also applicable to all waves initiated in the atmosphere by external impulses, which are subject to the fundamental equation of phase-velocity 7.4.

The discussion given in article 38, has shown that a group of waves initiated by an outside impulse would soon lose a great deal of its energy. The energy being dispersed and spread on the medium. The amplitude of the group would therefore be reduced to a small fraction of its initial value. Thus a group which may have an amplitude of 6000 km. one second after the impulse acted, will have an amplitude of only 100 km.
after a period of one hour. In 10 hours its amplitude will be reduced to 30 km. This results from consideration of the dispersive effect only. If friction is further accounted for, the amplitude after 10 hours would be reduced to 25 km.

Such waves are of course hardly detectable on the synoptic maps, for they simply appear as straight isobars approximating the features of undisturbed atmosphere. Assuming that the wave-length associated with the packet of waves in this particular example is 5000 km., and assuming that it moves with the high velocity of 50 km. per hour; it is seen that by the time it completes one period it has already been dissipated and become undetectable. If no more impulses are effective after the first one has ceased to act, the whole phenomenon will be similar to a pulse of light which is seen for a short period of time after which it is no longer visible. The pulse of light has only disappeared because its source was extinguished, and yet no one can claim that light is not a periodic motion. If the pulse of light consisted of one wave-length only, the analogy with atmospheric disturbances would be perfect.

Now, let it be assumed that the source of light is, for some reason, being flashed in an irregular manner. There will be no regular periods detectable in this case either, and any effort to find a periodicity in the times of observation of the separate pulses is condemned to fail, simply because of the reason that the external force which controls the source is non-periodic. But light itself will still
remain periodic.

A wave - cyclone is a packet of waves which is quickly dissipated by the atmosphere if the external force which initiated it ceases. As the external force which causes the formation of this packet is non-periodic by its nature, no periodicity will be observed. Meteorologists can only observe a packet as a whole and cannot observe its individual components, and the packet is continually being renewed by the outside non-periodical sources; and therefore it is not hoped that they will be able to detect any periodicity in the occurrences of these packets.

One point which remains vague so far is this: If the atmosphere is a very powerful dispersive medium in the way it was described above, then how is it possible for the same cyclonic system to persist for a long time, as it is usually observed to do? The answer to this question may be found in the hypothesis that the source of energy which gives rise to the impulses necessary for the maintenance of the system is not stationary, but it moves with the cyclonic system itself. And once this source of energy is exhausted the system loses the necessary impulses and thus dissipates quickly. In the case of the cyclones of middle latitudes, this source of energy may be potential, caused by the difference in densities of the interacting air masses; or it may be thermal caused by the solenoidal field arising from the contrast in temperatures of the air masses; or it may be internal in the form of latent energy of water vapor.
The solenoidal fields at the eastern coasts of the continents are stationary. They can only produce packets which die away rapidly; and the total effect of these fields of energy will appear as stationary troughs in their regions. Whereas the real thing that is happening is that troughs are being continually created at these localities and dissipated after they travel a short distance. It is these stationary troughs that are most effective in bringing air masses from different latitudes into juxtaposition, and thus cause the travelling sources of energy with which the wave cyclones are associated.

The stationary troughs are stationary in a dynamic sense and not in a static sense. Their equilibrium is similar to the equilibrium of table-clouds, or to the equilibrium of water vapor in a saturated closed container.
CHAPTER VII

WAVES AT A ZONE OF TRANSITION BETWEEN TWO LAYERS
OF UNIFORM MOTION

41- In his paper on "The Effect of Gradual Wind Change on The Stability of Waves", (Ref. 9), Professor Haurwitz treated the problem of sinuous waves at the zone of transition between two layers of uniform motion. The problem may be taken to represent waves at frontal surfaces, for example. In the following, Haurwitz' paper is summarized, and then extended in the light of the theory of group-velocity.

42- Assume a horizontal distribution of two fluids as shown in Fig. 8. The first fluid consists of layer I, which extends from \( y = 0 \) to \( y = -\infty \); the wind velocity in this layer being uniformly \( U_1 \). The second fluid consists of layer III, which extends from \( y = d \) to \( y = \infty \); the wind velocity in this layer being uniformly \( U_2 \). The zone of transition II, extends from \( y = 0 \) to \( y = d \). The wind velocity \( U_1 \), in this layer may be assumed to vary linearly from \( U_1 \) to \( U_2 \). The density is assumed to be the same in all layers. It is further assumed that the wave motion occurs in a horizontal
plane; and the vertical vertical component of the earth's rotation alone is taken into account.

The following two relations hold for the undisturbed motion in each layer:

\[ \ell V = -\frac{1}{\rho} \frac{\partial P}{\partial y} \]  
\[ J = -\frac{1}{\rho} \frac{\partial P}{\partial z} \]  

where \( \ell = \omega \sin \varphi \), \( g \) the acceleration of gravity, \( \rho \) the density, and \( P \) and \( U \) are the pressure and velocity of the undisturbed motion. The \( x \)-axis being taken to coincide with the boundary between the first layer and the layer of transition, and the \( z \)-axis vertically upwards.

The velocity in the transitional layer may be:

\[ U = U_0 + by \quad 0 < y < d \]  
\[ b = \frac{U_i - U_0}{d} \]

For the equilibrium pressure in layer II the following expression is found from equations 1.7, 2.7 and 3.7:

\[ P = \rho_0 - \rho \alpha \left( U_i y + \frac{1}{2} by^2 \right) - g \rho z \]  

The pressure in the three layers are therefore:

\[ P_1 = \rho_0 - \rho \alpha U_0 y - g \rho z \]  
\[ P_2 = \rho_0 - \rho \alpha \left( U_i y + \frac{1}{2} by^2 \right) - g \rho z \]  
\[ P_3 = \rho_0 - \rho \alpha \left( U_i y - \frac{1}{2} bd^2 \right) - g \rho z \]
The free upper surface of the fluid system is given by the condition:

\[ P = \text{Const.} \]

Denoting the height of the free surface by \( h \), it follows from 4.7a that for layer II:

\[ h = H - \frac{E}{g} \left( v y + \frac{1}{2} b y^2 \right) \quad \ldots \ldots \ldots \quad (5.7) \]

\( H \) being the height of the free surface where \( y = 0 \).

If \( u, v \) and \( p \) stand for the components of the perturbation velocity and the perturbation pressure, the equations of motion become, neglecting terms of second order of magnitude, :

\[
\frac{\partial u}{\partial t} + v \frac{\partial u}{\partial x} + b v - \ell v = - \frac{1}{\rho} \frac{\partial p}{\partial x}, \quad \ldots \ldots \ldots \quad (6.7)
\]

\[
\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} + \ell u = - \frac{1}{\rho} \frac{\partial p}{\partial y}, \quad \ldots \ldots \ldots \quad (6.7)
\]

And the equation of continuity becomes:

\[
\frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0
\]

or, approximately, :

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \ldots \ldots \ldots \quad (7.7)
\]

In the case of sinuous waves, a solution may be assumed in the form:

\[ u, v, p \propto e^{i(my-x \omega t)} \quad \ldots \ldots \ldots \quad (8.7) \]
where: \( m = \frac{aH}{X} \) and \( n = \frac{2\pi}{L} \).

From 8.7 and 7.7, we get:

\[
u = \frac{i}{m} \frac{d\nu}{dy} \quad \ldots \ldots \quad (9.7)
\]

And substituting in 8.7, we get:

\[
\frac{P}{\rho} = i\left( \frac{n - m\nu}{m} \right) \frac{d\nu}{dy} + \frac{i}{m}(b - \ell)\nu \quad \ldots \ldots \quad (10.7)
\]

\[
\text{and} \quad \frac{d\nu}{dy} - m\nu = 0
\]

The solution of this differential equation is:

\[
\nu = \mathcal{K}_1 e^{m\gamma} + \mathcal{K}_2 e^{-m\gamma} \quad \ldots \ldots \quad (11.7)
\]

The boundary conditions are: for layer I, \( \nu \neq \infty \)
when \( y \to -\infty \); and for layer III, \( \nu \neq \infty \) when \( y \to \infty \). These

give for layer I:

\[
\nu_i = \mathcal{K}_1 e^{m\gamma}
\]

\[
\frac{P}{\rho} = \left( \frac{n - m\nu}{m} \right) \mathcal{K}_1 e^{-m\gamma} \quad \ldots \ldots \quad (12.7)
\]

and for layer III

\[
\nu_s = \mathcal{K}_2 e^{-m\gamma}
\]

\[
\frac{P}{\rho} = -i\left( \frac{n - m\nu}{m} \right) \mathcal{K}_2 e^{m\gamma} \quad \ldots \ldots \quad (13.7)
\]

and for the transitional layer:

\[
\nu_z = \mathcal{K} e^{m\gamma} + \mathcal{C} e^{-m\gamma}
\]
\[ \frac{\rho}{\beta} = e^{-m_2 \lambda} \left( \lambda e^{-m_1 \lambda} - \lambda e^{-m_2 \lambda} \right) + \sum_{m} \left( \frac{b \ell}{m} \right) \left( \lambda e^{-m_1 \lambda} + \lambda e^{-m_2 \lambda} \right). \quad (14.7) \]

From the continuity of the velocity and the pressure at the boundaries of the transitional layer, it follows that:

\[ K_a = K_a + \lambda e^{-m_2 \lambda}, \quad \ldots \ldots \quad (15.7) \]

\[ K_a = K_a + \lambda e^{-m_2 \lambda}, \quad \ldots \ldots \quad (15.7) \]

and:

\[ b = \frac{1}{2} \left[ 2(n - m \lambda) - b \right] \lambda = 0, \]

\[ \left[ 2(n - m \lambda \lambda) + b \right] K_a e^{-m_2 \lambda} + b \lambda e^{-m_2 \lambda} = 0 \quad \ldots \ldots \quad (16.7) \]

from which, we get for the phase-velocity of the sinusoidal waves under consideration:

\[ V = \frac{\omega}{\beta}, \]

\[ \therefore \left( V - V_a \right) \left( V - V_a \right) + \frac{b \lambda}{H \Pi} \left( V_a - V_a \right) - \frac{b \lambda^2}{8 H \Pi} \frac{f}{1 + \epsilon \Pi \lambda} = 0 \]

And since:

\[ b \lambda = U_a - U_a, \]

we get from the last equation:

\[ V = \frac{V_a + V_a}{2} + \frac{V_a - V_a}{2} \sqrt{1 - \frac{\lambda}{H d} + \frac{\lambda^2}{d 2H \left(1 + \epsilon \Pi \lambda \frac{2H d}{H} \right)}}, \quad (17.7) \]

In the case of a thin layer of transition, we have:
\[ c \approx \frac{\pi d}{\lambda} \approx \frac{\lambda}{2 \pi d} \]

Substituting this in 17.7, we get:

\[ V = \frac{V_x + V_y}{2} + i \frac{V_x - V_y}{2} \sqrt{1 - \frac{2 \pi d}{\lambda}} \sqrt{1 + \frac{2 \pi d}{\lambda}} \quad (18.7) \]

And when \( d = 0 \), we get for an abrupt transitional layer:

\[ V = \frac{V_x + V_y}{2} + i \frac{V_x - V_y}{2} \quad (19.7) \]

43- Group - Velocity of Shearing Waves

As in the previous similar cases, equation 17.7 is only valid when we have simple sinuous trains of waves progressing at the layer of transition between two layers of fluid flowing parallel to each other in the way described in the last article. In order that 17.7 might be applied to actual cases as they happen in the atmosphere, we need further to assume that those waves do not interfere with any other waves at the same surface where they are being propagated.

It is obvious that such conditions are not realized in the atmosphere. The waves which are observed at the frontal surfaces are far from being sinusoidal. They can best be assumed to have the general form of waves initiated by momentary impulses at a single narrow region in the layer where they appear as travelling disturbances. It is also the case that the original source of energy which initiates the frontal waves does not cease after the first impulse, but it continues
in action for a long time, thus initiating other disturbances on the same layer. These disturbances interfere with the wave system of the first impulse and thus create new groups of waves in a continuous manner.

To compute the group-velocity of frontal waves, use has to be made of the Kelvin-Green's theory of group-velocity. The reasoning outlined in the last chapter shows that a frontal wave does not have a definite group-velocity because we are dealing with a continuous source of impulses. That is to say, that as long as the source of energy which creates the necessary impulses for the maintenance of the disturbance, is moving with the disturbance, and as long as these impulses give rise to new groups of waves in a more or less continuous manner, then there will be no definite group-velocity. The frontal systems which are ordinarily observed on the synoptic maps are continuously undertaking a process of being created and annihilated. They are, therefore, not constituted of the same group of waves for which a formula might be computed, but they are constituted of a collection of groups. The individual groups are only controlled by the behavior of the source of energy, and the way it is supplying the individual impulses.

For the sake of completeness, the group-velocity of shearing waves will here be computed, and a brief discussion of the resulting equations will be given.

To make the mathematical analysis legitimate, it
is assumed that a certain definite point on the initial group is being followed, a maximum for example. It is further assumed that a certain wave-length is associated with that point, the initial wave-length of the group for example. And it is required to compute the velocity of that definite point where that definite wave-length will always be observed.

Under these assumptions, equation 10.5 may be used to get the group-velocity. Thus:

$$W = f(m) + mf'(m) \quad (10.5)$$

And substituting from 17.7, we get:

$$W = \sqrt{\frac{\lambda}{\pi d} + \frac{\lambda^2}{d^2 \pi^2 \left(1 + \frac{2 \pi d}{\lambda} \coth \frac{2 \pi d}{\lambda}\right)^2}}$$

or

$$W = \frac{W^0 + V}{2} + \frac{W^0 - V}{2} \sqrt{1 - \frac{\lambda}{\pi d} + \frac{\lambda^2}{d^2 \pi^2 \left(1 + \frac{2 \pi d}{\lambda} \coth \frac{2 \pi d}{\lambda}\right)^2}}$$

$$= \frac{W^0 - V}{2} \frac{\left(\frac{\lambda}{\pi d} + \frac{\lambda^2 \coth \frac{2 \pi d}{\lambda}}{d^2 \pi^2 \left(1 + \frac{2 \pi d}{\lambda} \coth \frac{2 \pi d}{\lambda}\right)^2}\right) + \frac{\lambda^2}{d^2 \pi^2 \left(1 + \frac{2 \pi d}{\lambda} \coth \frac{2 \pi d}{\lambda}\right)^2}}{\sqrt{1 - \frac{\lambda}{\pi d} + \frac{\lambda^2}{2 \pi^2 d^2 \left(1 + \frac{2 \pi d}{\lambda} \coth \frac{2 \pi d}{\lambda}\right)^2}}} \quad (20.7)$$

Equation 20.7 gives the group-velocity for shearing waves at a transitional layer of any thickness $d$.

For the case of a thin layer of transition, either 20.7 or 18.7 may be used, from which we get:
\[ W = V' - i \frac{V_e - V_i}{2} \sqrt{1 - \frac{2 \pi d}{\lambda}} \left(1 + \frac{4 \pi d}{\lambda} \right) \sqrt{1 - \frac{4 \pi d^2}{\lambda^2}} \] 

\[ W = \frac{V_e + V_i}{2} + i \frac{V_e - V_i}{2} \sqrt{\frac{1 - \frac{2 \pi d}{\lambda}}{1 + \frac{4 \pi d}{\lambda}}} - i \frac{2V_e - V_i}{2} \frac{\frac{2 \pi d}{\lambda}}{\left(1 + \frac{2 \pi d}{\lambda} \right) \sqrt{1 - \frac{4 \pi d^2}{\lambda^2}}} \] 

\[ W = \frac{V_e + V_i}{2} + i \frac{V_e - V_i}{2} \sqrt{\frac{1 - \frac{2 \pi d}{\lambda}}{1 + \frac{4 \pi d}{\lambda}}} - \frac{\frac{2 \pi d}{\lambda}}{\left(1 + \frac{2 \pi d}{\lambda} \right) \sqrt{1 - \frac{4 \pi d^2}{\lambda^2}}} \] 

(21.7)

In the case of waves at an abrupt layer of transition, it is seen from equation 19.7 that the phase-velocity does not depend on the wave-length. Therefore the medium does not act as a dispersive medium with respect to such waves. The group velocity will, therefore, be equal to the phase-velocity. Thus we have, for this case:

\[ W = V = \frac{V_e + V_i}{2} + i \frac{V_e - V_i}{2} \] 

(22.7)

44- The Stability of Groups of Waves Under The Effect of a Gradual Wind Change

Mathematically, a wave system is unstable when its velocity contains an imaginary term. For then the system as a whole will not only propagate in the direction of the
of the surface at which it is initiated, but will also move in a direction perpendicular to that surface, a motion which tends to increase the amplitude of the system continuously.

Going back to equation 20.7, we find that the group-velocity, in the general case of an arbitrary thickness of the transitional layer, will contain an imaginary term if:

$$d \frac{2 \pi}{\lambda} \left( \frac{d}{\lambda} - 1 \right) \left( 1 + \coth \frac{2 \pi}{\lambda} d \right) + 1 < 0$$

This condition is satisfied when:

$$\frac{2 \pi}{\lambda} d \leq 1.2785$$

or when:

$$d \leq 0.204 \lambda \quad \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots \cdots (23.7)$$

The condition for the instability of a group of waves on a frontal surface is therefore the same as that for the instability of a single sinusoidal train of waves on the same surface. The latter case has been discussed fully in the paper of Prof. Haurwitz to which reference has been made. And it is sufficient here to refer the reader to the same paper. All the results which were found to be valid for the instability of simple sinusoidal shearing waves are also valid for the instability of a group of shearing waves.

It might be mentioned, however, that the general discussion given in the last two chapters has shown that as a packet of waves progresses along the disturbed surface, it continuously gains in wave-length, and loses in amplitude.
From equation 23.7, it is readily seen that any increase in wave-length tends to make that condition easier to satisfy. That is it tends to make the group more unstable. Due to this factor alone, the dispersive effect of the medium, therefore, acts to decrease the stability of the system.

Considering the physical situation of what is happening to an unstable packet of waves, we get a different picture. For it has been mentioned at the beginning of this article that in an unstable wave the amplitude increases in a continuous manner. From equation 9.5, it is seen that the amplitude of a packet of waves decreases continuously with time. This shows that, due to this factor alone, dispersion is a stabilizing factor. It was shown in the last chapter, also that the curve of the amplitude of a packet of waves falls very rapidly in the early stages of its life, after which it levels off and starts to decrease very slowly. Accordingly, dispersion is seen to be a very powerful stabilizing factor in the early life of a disturbance. The effect of dispersion in lengthening the wave-length of the whole packet is never so powerful; and as a matter of fact it could be it—could—be neglected in comparison with the effect of amplitude, in the early stages of the disturbance. In the later stages the reverse is true. The amplitude effect becomes very slow, and the wave-length effect became greater. However, in the later stages, a disturbance would have been almost lost from the weather map, if it does not renew its energy by being acted upon by some new impulses.
Fig.

- Dispersion curve (vertical scale magnified 50 times)
- Friction curve
- Total curve (vertical scale magnified 50 times)
From this, it can be concluded that the main effect of dispersion on shearing waves is to reduce their amplitudes, and thus to act as a stabilizing factor; and by that it will be hindering the de-stabilizing effect of the shearing forces.

Since the dispersion effect is exceedingly powerful in the first few hours of the life of the disturbance, as seen from Fig.9, it can safely be stated that all such disturbances should be stable during that period.

It remains to investigate the effect of the imaginary term in the expression for velocity, and how it compares with that of dispersion.

It is easily shown that the rate of increase of the amplitude of an unstable wave varies exponentially with time. Thus in the case which we are considering:

$$e^{rt}$$  (See Ref.24)

where \( r \) is a function of the imaginary part of the velocity as given from equation 20.7.

The curve for this variation is given in Fig.9. It is seen from this curve that the exponential factor is less powerful than the dispersion factor in the first stages of the life of the disturbance. The relative magnitude of the two can only be compared when the quantitative magnitude of \( r \) is known; But no matter how great it may be, the whole effect will still remain smaller than that of the dispersion factor.
This is especially true in the first few hours of the life of the disturbance. But after a reasonably enough time, the shearing effect will be more powerful, and the wave may become unstable, when the shearing effect overcomes that of dispersion.

It seems to this writer that the stabilizing effect of dispersion has been neglected in earlier treatises on this subject. And it might safely be stated that it provides an explanation for the observed fact that waves are usually stable at their initial stages.

If the degree of instability is measured by the rate at which the amplitude increases, then we may say:

Degree of instability due to shearing effect \( \sim e^{rt} \) \( (24.1) \)

and, degree of instability due to dispersion effect \( \sim \frac{A^2}{k^2} \) \( (25.7) \)

If \( t \) is small, 24.7 can be expanded by a Taylor's series, and thus we get:

Degree of instability due to shearing effect

\[ \sim 1 + (rt) + \frac{(rt)^2}{L^2} + \frac{(rt)^3}{L^3} + \ldots \] \( (24.7a) \)

Taking the first two terms of the series, and comparing with 25.7, we see that the stabilizing power of dispersion is greater than the destabilizing power of shear.

If \( t \) is big, the terms of second and higher degrees can no more be neglected; for in fact they become much more predominant than the linear terms, and it is readily seen
that when \( t \) is big the shearing effect becomes more powerful than the dispersion effect.

This result agrees with that derived from the qualitative discussion which was based on the shape of the curves of Fig. 9, and no further comments are felt necessary.
CHAPTER VIII

GROUP - VELOCITY IN A COMpressible FLUID

45- Lagrangge's Hydrodynamical Equations

The Lagrange's form of the hydrodynamical equations is best suited for the discussion of motion in compressible fluids as given in the present chapter. We start, therefore, by writing them down. (Ref. 25).

Let a, b, c be the initial coordinates of a particle, and x, y, z the coordinates of the same particle at time t, then a, b, c, t are the independent variables and our object is to determine x, y, z in terms of a, b, c, t and so investigate completely the motion. Let it be assumed that there exists a potential Φ for the external forces. The equations of motion will thus be:

\[ \frac{\partial x}{\partial t} + \frac{\partial y}{\partial t} \frac{\partial y}{\partial a} + \frac{\partial z}{\partial t} \frac{\partial z}{\partial a} = -\frac{\partial \mathcal{A}}{\partial a} - \frac{1}{\rho} \frac{\partial p}{\partial a}, \quad \ldots \quad (a) \]

\[ \frac{\partial x}{\partial t} + \frac{\partial y}{\partial t} \frac{\partial y}{\partial b} + \frac{\partial z}{\partial t} \frac{\partial z}{\partial b} = -\frac{\partial \mathcal{A}}{\partial b} - \frac{1}{\rho} \frac{\partial p}{\partial b}, \quad \ldots \quad (b) \]

\[ \frac{\partial x}{\partial t} + \frac{\partial y}{\partial t} \frac{\partial y}{\partial c} + \frac{\partial z}{\partial t} \frac{\partial z}{\partial c} = -\frac{\partial \mathcal{A}}{\partial c} - \frac{1}{\rho} \frac{\partial p}{\partial c}, \quad \ldots \quad (c) \]

And the equation of continuity:

\[ Q \frac{\partial (x, y, z)}{\partial (a, b, c)} = 0 \quad \ldots \quad (2.8) \]
In the case of an incompressible fluid, the equation of continuity takes the form:

\[
\frac{\partial (x, y, z)}{\partial (a, b, c)} = 1 \quad \cdots \quad \cdots \quad \cdots \quad (3.8)
\]

For perurbation motion, the equations of motion become: (Ref. 1),

\[
\begin{align*}
\frac{\partial x}{\partial t} & \frac{\partial x}{\partial a} + \frac{\partial y}{\partial t} \frac{\partial y}{\partial a} + \frac{\partial z}{\partial t} \frac{\partial z}{\partial a} + \frac{\partial x}{\partial t} \frac{\partial x}{\partial a} + \frac{\partial y}{\partial t} \frac{\partial y}{\partial a} + \frac{\partial z}{\partial t} \frac{\partial z}{\partial a} \\
& \quad + \frac{\partial}{\partial a} \left[ \frac{p}{\partial x} + \frac{\partial q}{\partial y} \frac{\partial x}{\partial y} + \frac{\partial q}{\partial z} \frac{\partial x}{\partial z} \right] + \left( \frac{\partial q}{\partial a} - \gamma \frac{\partial p}{\partial a} \right) \frac{p}{\partial a} = 0 \\
\frac{\partial x}{\partial t} & \frac{\partial x}{\partial b} + \frac{\partial y}{\partial t} \frac{\partial y}{\partial b} + \frac{\partial z}{\partial t} \frac{\partial z}{\partial b} + \frac{\partial x}{\partial t} \frac{\partial x}{\partial b} + \frac{\partial y}{\partial t} \frac{\partial y}{\partial b} + \frac{\partial z}{\partial t} \frac{\partial z}{\partial b} \\
& \quad + \frac{\partial}{\partial b} \left[ \frac{p}{\partial x} + \frac{\partial q}{\partial y} \frac{\partial x}{\partial y} + \frac{\partial q}{\partial z} \frac{\partial x}{\partial z} \right] + \left( \frac{\partial q}{\partial b} - \gamma \frac{\partial p}{\partial b} \right) \frac{p}{\partial b} = 0 \\
\frac{\partial x}{\partial t} & \frac{\partial x}{\partial c} + \frac{\partial y}{\partial t} \frac{\partial y}{\partial c} + \frac{\partial z}{\partial t} \frac{\partial z}{\partial c} + \frac{\partial x}{\partial t} \frac{\partial x}{\partial c} + \frac{\partial y}{\partial t} \frac{\partial y}{\partial c} + \frac{\partial z}{\partial t} \frac{\partial z}{\partial c} \\
& \quad + \frac{\partial}{\partial c} \left[ \frac{p}{\partial x} + \frac{\partial q}{\partial y} \frac{\partial x}{\partial y} + \frac{\partial q}{\partial z} \frac{\partial x}{\partial z} \right] + \left( \frac{\partial q}{\partial c} - \gamma \frac{\partial p}{\partial c} \right) \frac{p}{\partial c} = 0
\end{align*}
\]

and the equation of continuity becomes:

\[
\begin{bmatrix}
\frac{\partial X}{\partial a} \\
\frac{\partial X}{\partial b} \\
\frac{\partial X}{\partial c}
\end{bmatrix}
\begin{bmatrix}
p \\
\frac{\partial Q}{\partial a} \\
\frac{\partial Q}{\partial b}
\end{bmatrix}
+ \begin{bmatrix}
\frac{\partial Y}{\partial a} \\
\frac{\partial Y}{\partial b} \\
\frac{\partial Y}{\partial c}
\end{bmatrix}
\begin{bmatrix}
p \\
\frac{\partial Q}{\partial a} \\
\frac{\partial Q}{\partial b}
\end{bmatrix}
+ \begin{bmatrix}
\frac{\partial Z}{\partial a} \\
\frac{\partial Z}{\partial b} \\
\frac{\partial Z}{\partial c}
\end{bmatrix}
\begin{bmatrix}
p \\
\frac{\partial Q}{\partial a} \\
\frac{\partial Q}{\partial b}
\end{bmatrix}
= 0 \quad (*.8)
\]

\[
y = \gamma \rho \quad \cdots \quad \cdots \quad \cdots \quad (6.1)
\]
Where $P$ is the pressure of the undisturbed fluid, $p$ is the perturbation pressure, $Q$ is the density of the undisturbed fluid, $q$ is the perturbation component of density, and $\gamma$ is the coefficient of baratropy.

For the undisturbed motion, equations 1.8 and 2.8 are valid as they are, after the substitution of $P$ for $p$.

46- Wave Motion at an Internal Surface of Discontinuity

We shall consider wave motion at the internal surface of discontinuity between two isothermal compressible fluids. The surface of discontinuity may be assumed horizontal at $c = 0$, and both layers may be assumed infinite in depth, so that the lower layer extends toward $c = -\infty$, and the upper layer toward $c = \infty$. The effect of the earth's rotation will be neglected.

Assume that:

$$Q = Q(c)$$

Hence

$$\frac{\partial Q}{\partial c} = \frac{dQ}{dP} \frac{dP}{dc} = \rho \frac{dP}{dc}$$

where $\gamma$ is the coefficient of baratropy. And from the equation of state, we have:

$$Q = \frac{P}{\gamma R}$$

Also from Poisson's equation, we have:

$$\frac{P}{P_0} = \left(\frac{\eta_0 - \alpha c}{\eta_0}\right)^{\frac{\gamma}{\gamma - 1}}$$

where $\alpha$ is the lapse rate of temperature.
Therefore

\[ r' = \frac{dQ}{dp} = \left(1 - \frac{R\alpha}{\nu} \right)^{-1} \frac{1}{R(\nu - \alpha)} \]

Assuming adiabatic conditions, we get for the total motion:

\[ \bar{p} \bar{q} - \frac{c_r}{\nu} = \text{const} \]

\[ \therefore \quad \gamma = \frac{d\psi}{dp} = \frac{c_r}{\nu \nu' \nu''} = \frac{1}{\kappa R \nu' \nu''} \]

The equations of motion for perturbation motion in two dimensions become:

\[ \frac{\partial^2 x}{\partial t^2} + \frac{\partial}{\partial a} \left( \frac{r}{Q} + \nu \frac{\partial \nu}{\partial a} \right) = 0, \ldots \ldots \ldots (7.8a) \]

\[ \frac{\partial^2 \nu}{\partial t^2} + \frac{\partial}{\partial c} \left( \frac{r}{Q} + \nu \frac{\partial \nu}{\partial c} \right) - \left( r' - \nu \right) g \frac{r}{Q} = 0 \quad \ldots \ldots (7.8b) \]

and the equation of continuity:

\[ \gamma \frac{r}{Q} + \frac{\partial x}{\partial a} + \frac{\partial \nu}{\partial c} = \gamma \quad \ldots \ldots (7.8c) \]

Since \( Q = Q(a) \), 7.8a and 7.8b become:

\[ \frac{\partial^2 x}{\partial a^2} + \frac{1}{\alpha} \frac{\partial^2 r}{\partial a^2} + \nu \frac{\partial^2 \nu}{\partial a^2} = 0, \alpha \nu \quad \ldots \ldots (8.8a) \]

\[ \frac{\partial^2 \nu}{\partial c^2} + \frac{1}{\alpha} \frac{\partial^2 r}{\partial c^2} + \nu \frac{\partial^2 \nu}{\partial c^2} + r g \frac{\partial^2 r}{\partial a^2} = 0 \quad \ldots \ldots (8.8b) \]

Assume a solution in the form:
\[ \kappa = \alpha(v) e^{i[m_a - (n-mu)v]t}, \]
\[ \gamma = \beta(v) e^{i[m_a - (n-mu)v]t}, \]
\[ \rho = \gamma(v) e^{i[m_a - (n-mu)v]t}, \]

where \( U \) is the undisturbed velocity of the fluid, assumed in the \( a \)-direction.

Substituting in 8.8, we get:

\[
\begin{cases}
-(n-mu)^2 \dot{A} + \frac{i}{Q} \dot{m} \dot{D} + \dot{g} \dot{m} \dot{E} = 0, \\
-(n-mu)^2 \dot{E} + \frac{i}{Q} \frac{dD}{dc} + \frac{g}{Q} \frac{dE}{dc} + \frac{g}{Q} \dot{D} = 0
\end{cases}
\] (9.8)

And from the equation of continuity, we get:

\[
-f \frac{D}{Q} = \dot{m} A + \frac{dE}{dc} \] (9.9)

From 9.8a, we get:

\[ A = \dot{m} \frac{\frac{D}{Q} + g \dot{E}}{(n-mu)^2}. \]

And substituting in 9.8c, we get:

\[
(n-mu)^2 \dot{E} - m \ddot{g} \dot{E} = \left[ m' - Y (n-mu) \right] \frac{D}{Q} \] (10.8)

From 9.8b, we get:

\[
g \dot{E} - (n-mu) \dot{E} = gY \frac{D}{Q} - \frac{D'}{Q} \] (11.8)
Eliminating $\xi'$ between 10.8 and 11.8, we get after some reductions:

\[
\xi' = \frac{(n - m\nu)D'' + \left[m + r'(n - m\nu)\right]gD + r_mg'D}{Q\left[(n - m\nu)' - m\nu'\right]} \quad (12.8)
\]

Substituting this in 11.8, we get:

\[
D'' + y_rD' + \left[\frac{m\nu'}{(n - m\nu)'} - m\nu' + \frac{r_m\nu'}{(n - m\nu)'}\right]D = 0 \quad (13.8)
\]

To solve this differential equation, we assume

\[
D \sim \xi c
\]

Substituting in 13.8, and solving for $s$, we get:

\[
\xi = -\frac{\xi c'}{2} + N \quad \ldots \ldots \ldots \ldots \ldots \quad (14.8)
\]

where:

\[
N = \sqrt{\frac{\xi c'}{2} + m^2 - r(n - m\nu) - \left(r^2 - 1\right) \frac{m\nu'}{(n - m\nu)'}} \quad (15.8)
\]

Therefore for the lower layer, we get:

\[
D = \xi' \xi c + \xi c' + N^c
\]

\[
\therefore \quad P^c = \left[\xi' \xi c + N^c\right] \xi \left[\xi c - (n - m\nu)c\right]
\]

And for the upper layer, we get a similar expression.

From the boundary conditions, for the upper layer when $c \to \infty$, $\xi \neq \infty$, and for the lower layer when $c \to \infty$, $\xi \neq \infty$, we finally get:
\[ P^I = \kappa^I \alpha^{-\gamma \frac{e'}{2} L + \beta \gamma \frac{e'}{2} L} \frac{i^{1-(n+m^I)}}{L} \quad \text{and} \]

\[ P^\Pi = \kappa^\Pi \alpha^{-\gamma \frac{e''}{2} L - \beta \gamma \frac{e''}{2} L} \frac{i^{1-(n+m^\Pi)}}{L} \]

\[ D^I = \kappa^I \alpha^{-\gamma \frac{e'}{2} L + \beta \gamma \frac{e'}{2} L} \]

\[ D^\Pi = \kappa^\Pi \alpha^{-\gamma \frac{e''}{2} L - \beta \gamma \frac{e''}{2} L} \]

\[ \mathcal{E}^I = \frac{(n+m^I)(-\gamma \frac{e'}{2} L + N^I)}{K^I \alpha^{-\gamma \frac{e'}{2} L + \beta \gamma \frac{e'}{2} L} + m^I \gamma \frac{e'}{2} L - \frac{1}{2} (n+m^I)}{Q^I \left[(n+m^I)^2 - \gamma m^I \right]} \quad \text{and} \]

\[ \mathcal{E}^\Pi = \frac{(n+m^\Pi)(-\gamma \frac{e''}{2} L - N^\Pi)}{K^\Pi \alpha^{-\gamma \frac{e''}{2} L - \beta \gamma \frac{e''}{2} L} + m^\Pi \gamma \frac{e''}{2} L - \frac{1}{2} (n+m^\Pi)}{Q^\Pi \left[(n+m^\Pi)^2 - \gamma m^\Pi \right]} \]

At the boundary: \( c = 0 \)

\[ P^I_0 = P^\Pi_0 \quad ; \quad i^I = i^\Pi \quad ; \quad \beta = \beta \]

\[ D^I_0 = D^\Pi_0 \quad ; \quad K^I = K^\Pi \quad ; \quad \mathcal{E}^I_0 = \mathcal{E}^\Pi_0 \]

\[ \frac{(n+m^I)(-\gamma \frac{e'}{2} L + N^I) + \gamma m^I}{Q^I_0 \left[(n+m^I)^2 - \gamma m^I \right]} = \frac{(n+m^\Pi)(-\gamma \frac{e''}{2} L - N^\Pi) + \gamma m^\Pi}{Q^\Pi_0 \left[(n+m^\Pi)^2 - \gamma m^\Pi \right]} \quad \text{... (16.8)} \]

Let: \( U^I = U^\Pi = 0 \) \quad \text{... 16.8 becomes:}

\[ \frac{m^I + \gamma m^I}{Q^I_0 \left(n^I - \gamma m^I \right)} = \frac{m^\Pi + \gamma m^\Pi}{Q^\Pi_0 \left(n^\Pi - \gamma m^\Pi \right)} \quad \text{... (17.8)} \]
This equation may be satisfied by:

\[ \eta^r - \delta^r m^r = 0 \quad \text{or} \]

\[ V^* = \sqrt{\frac{2\eta}{x^r}} \quad \text{(Stoke's velocity)} \]

This gives \( \xi = \infty \) or \( p = 0 \), and has to be rejected.

Equation 17.8 may also be satisfied if

\[ \frac{\eta^r \delta^{r'} + \delta^r m^{r'}}{Q^r_{,r}} = \frac{\delta^{r'} + \delta^{r'} m^{r'}}{Q^{r'}} \]

This condition reduces to:

\[ \frac{\eta^r \delta^{r'} + \delta^r m^{r'}}{\gamma^{r'}} = \frac{\delta^{r'} + \delta^{r'} m^{r'}}{\gamma^{r'}} = \sigma \]

\[ \delta^r = \frac{\gamma^{r'}}{m^r} \sigma - \frac{m^{r'}}{m^r} \quad \text{and} \quad \ldots \ldots \quad (17a) \]

\[ \delta^{r'} = \frac{\gamma^{r'}}{m^r} \sigma - \frac{m^{r'}}{m^r} \quad \ldots \quad (17b) \]

From 17.8, we get:

\[ \sigma^2 + L^r \sigma + M^r = 0 \quad \ldots \quad (18.8) \]

where:

\[ L^r = \delta^r \left( \eta^r - \frac{m^r}{\gamma^r} \right) \quad \text{and} \quad M^r = \frac{L^r}{\gamma^r} \left( \frac{m^r}{\gamma^r} - m^r \right) \]

And from 17.8b, we get:

\[ \sigma^2 + L^r \sigma + M^r = 0 \quad \ldots \quad (19.8) \]

where \( L^r \) and \( M^r \) have similar meanings to \( L^f \) and \( M^f \).
Solving 13.8 and 19.8, we get:

\[ \sigma^r = -\frac{M^r - M^d}{l^r - l^d} \quad \ldots \ldots \ldots \quad (20.8) \]

technically 18.8 becomes:

\[ (M^r - M^d) + (l^r - l^d)(L^r M^d - L^d M^r) = 0 \quad \ldots \ldots (21.8) \]

Putting:

\[ \frac{1}{y^r} + \frac{1}{y^d} = \kappa R (\gamma^r + \gamma^d) = 2s^r \]

where \( s \) is the velocity of sound,

and

\[ \frac{1}{y^r} - \frac{1}{y^d} = \kappa R (\gamma^r - \gamma^d) = 2r^r \]

so that:

\[ \frac{1}{y^r} \left( \frac{1}{y^r} - \frac{1}{y^d} \right) = s^r - r^r \]

we get from 21.8:

\[ (j^r m^r - n^r)(2s^r m^r - n^r) + 2j^r m^r \left[ \kappa n^r (2s^r - n^r) - 2m^r (s^r - r^r) \right] = 0 \quad \ldots (23.8) \]

Let:

\[ \chi = \frac{v^r}{s^r} = \frac{m^r}{m^r s^r} \quad \text{and} \]

\[ \omega = \frac{\gamma^r}{2s^r s^r} = \frac{g}{m^r s^r} \]

And substituting in 23.8, we get:

\[ (\omega^r - \chi^r)(2 - \chi^2) + 2\omega \left[ \kappa \chi (2 - \chi) - 2 + 2 \frac{r^r}{s^r} \right] = 0 \]

From which:

\[ \omega^r = \frac{\chi^r (\chi - 2)^r}{-(2\kappa - 1) \kappa + \kappa (\kappa - 1) \kappa + \frac{2}{s^r}} \quad \ldots \ldots (24.8) \]
where \[
\frac{r^2}{\lambda^2} = \left( \frac{\rho^2 - \rho^1}{\rho^2 + \rho^1} \right)^2
\]

And \[V = s \cdot x^\nu . \quad \ldots \ldots \ldots (25.8)\]

The last solution was taken from Haurwitz' paper "Zur Theorie der Wellenbewegungen in Luft und Wasser", (Ref.10), and from his paper "Über Wellenbewegungen an der Grenzfläche zweier Luftschichten mit lineare Temperaturgefälle".

47-Group - Velocity in Compressible Fluids

Equations 24 and 25.8 give the phase-velocity for a train of sinusoidal waves at the common surface between two isothermal layers of air, both of which have no horizontal motion. For the sake of simplicity, the following discussion will be restricted to this special case.

From equation 11.5, we have for the group-velocity:

\[\mathcal{W} = \mathcal{f}(\nu) + \nu \mathcal{f}(\nu) \quad \ldots \ldots \ldots (11.5)\]

Substituting from 24 and 25.8, we get, after some simple transformations:

\[
\mathcal{W} = s x^k \frac{\omega V^n}{2 \omega^k} \left( \frac{-(2 \nu - 1) x^2 + 2 (\nu - 1) x + \frac{4 \nu^2}{s^2}}{2 x^2 - 6 x^2 + 4 x + (2 \nu - 1)/x \omega^2 - 2 (\nu - 1)/\omega^2} \right) \ldots (26.8)
\]

where \(V^n\) is the Stokes' velocity given by:

\[V^n = \frac{g}{s} \]

Since \(x\) is a very small quantity compared with
unity, the second and higher powers will be neglected in the
following computations, and the following approximate formula
will be used:

\[ W = V - \frac{2V^2 \omega}{\kappa} \left( \frac{(2K-1)\omega}{\xi} + \frac{\nu}{\xi} \right) \left( \frac{(2K-1)\omega^2}{2} \right) \quad \cdots (27.8) \]

where \( V \) is the phase-velocity as given by equation 25.8.

The following table gives some values for the
group-velocity, the phase-velocity, and Stokes' velocity;
the group-velocity being computed from the approximatation formula
27.8. This table is to be compared with that given by Haarwitz
(Ref. 10). The following values were taken for the parameters
in formula 27.8:

- \( s \), the velocity of sound = 330 m/sec.
- \( k \), the ratio of specific heats of air = 1.405
- \( \frac{\nu}{s^2} = 10^{-2} \)

<table>
<thead>
<tr>
<th>Wavelength (( \lambda ))</th>
<th>Phase-Vel. (( V ))</th>
<th>Stokes' Vel. (( V* ))</th>
<th>Group-Vel. (( W ))</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.0m</td>
<td>0.33 m/sec.</td>
<td>0.33 m/sec.</td>
<td>-0.21 m/sec.</td>
</tr>
<tr>
<td>34.8</td>
<td>0.74</td>
<td>0.74</td>
<td>-0.51</td>
</tr>
<tr>
<td>69.2</td>
<td>1.05</td>
<td>1.04</td>
<td>-0.61</td>
</tr>
<tr>
<td>200</td>
<td>1.94</td>
<td>1.77</td>
<td>-1.07</td>
</tr>
<tr>
<td>436</td>
<td>3.22</td>
<td>2.61</td>
<td>-2.06</td>
</tr>
<tr>
<td>597</td>
<td>4.03</td>
<td>3.05</td>
<td>-3.00</td>
</tr>
<tr>
<td>1048</td>
<td>6.36</td>
<td>4.05</td>
<td>-22.95</td>
</tr>
</tbody>
</table>
The results are most interesting, since here have another case of negative group-velocity. The group as a whole travels in the opposite direction to that in which its components travel.

Going back to formula 27.8, we see, however, that the group-velocity is not always negative. It is negative only when the following condition is satisfied:

\[ \sqrt{\frac{2\omega V x^k (K-1) x + \frac{V^2}{2\omega}}{x^k + (2K-1) \omega^2 x^k - 2(K-1) \omega^2}} \quad \ldots \quad (27.8') \]

To simplify the discussion, the middle term in the denominator will be neglected; it is always small in comparison with the remaining two terms. Inequality 28.8' will thus be reduced to the following form:

\[ \sqrt{\frac{2\omega V x^k}{x^k} (K-1) x + \frac{V^2}{2\omega} \quad \ldots \quad (28.8) \]

In the right hand side of this inequality, all the variables are positive, if the positive sign of \( x^k \) is taken. \( x^k \) is however, always positive for positive values of \( V \). From this we see that the right hand side of the inequality is positive as long as the following condition is satisfied:

\[ 4x > 2(K-1) \omega^2 \]

But, when:

\[ 4x < 2(K-1) \omega^2 \quad \ldots \quad (28.8'') \]

the right hand side of the inequality will have a negative sign but its absolute value will be less than that of \( V \) as long as
the latter is positive. Therefore, if 28.8" is satisfied
the group-velocity would be positive and its absolute value
would be greater than that of the phase-velocity.

Substituting from 24.8 in 28.8", we see that in
order that the group-velocity may be greater than the phase-
velocity, the following condition must be satisfied:

\[ 4 \phi < \frac{2(k-1)}{(2(k-1))x^2 + 4(k-1)x + 4 \frac{\phi}{2}} \]

Or, neglecting terms of second order of magnitude,

\[ 4 \phi < \frac{2(k-1)}{4(k-1)x + 4 \frac{\phi}{2}} \text{ or } (k-1)x < -\frac{\phi}{2}. \]

But since \((k-1) > 0\), and \(x > 0\), the left hand side of the
inequality \(> 0\). And since the right hand side of the ineq-
uality is always negative, the condition as a whole is never
satisfied.

We conclude, therefore, that the group-velocity
can never be greater than the phase-velocity as long as they
are both in the same direction.

From formula 26.8, the condition which has to be
satisfied in order that the group-velocity will be equal to
the phase-velocity, is:

\[ -(2k-1)x^2 + 4(k-1)x + 4 \frac{\phi}{2} = 0, \phi \]

\[ \phi = \frac{2(k-1) \pm 2\sqrt{(k-1)^2 + \frac{\phi}{2}(2k-1)}}{2k-1} \]

And if the temperature difference between the two layers is
small, the term \(\phi(2k-1)\) may be neglected, and the last equation
reduces to:

\[ -(2k-1)x^2 + 4(k-1)x = 0 \]
which gives for \( x \):

\[ x = 0, \text{ or} \]

\[ x \approx \frac{4(k-1)}{2k-1} \approx 0.9 \]

This simply means that the wave-velocity should be comparable to the velocity of sound. It is a known fact (Art. 5) that the group-velocity of sound waves is equal to their phase-velocity; and it may be assumed that the last result would have been \( x = 1 \), were it not for the fact that the approximate formula was used for its derivation. The velocity of sound is known to be the maximum velocity that any wave can attain in the air. And it can safely be stated that no gravitational wave can ever attain the velocity of sound. Therefore it might be concluded that the group-velocity can never be equal to nor greater than the phase-velocity. It has to remain always less than the phase-velocity.
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